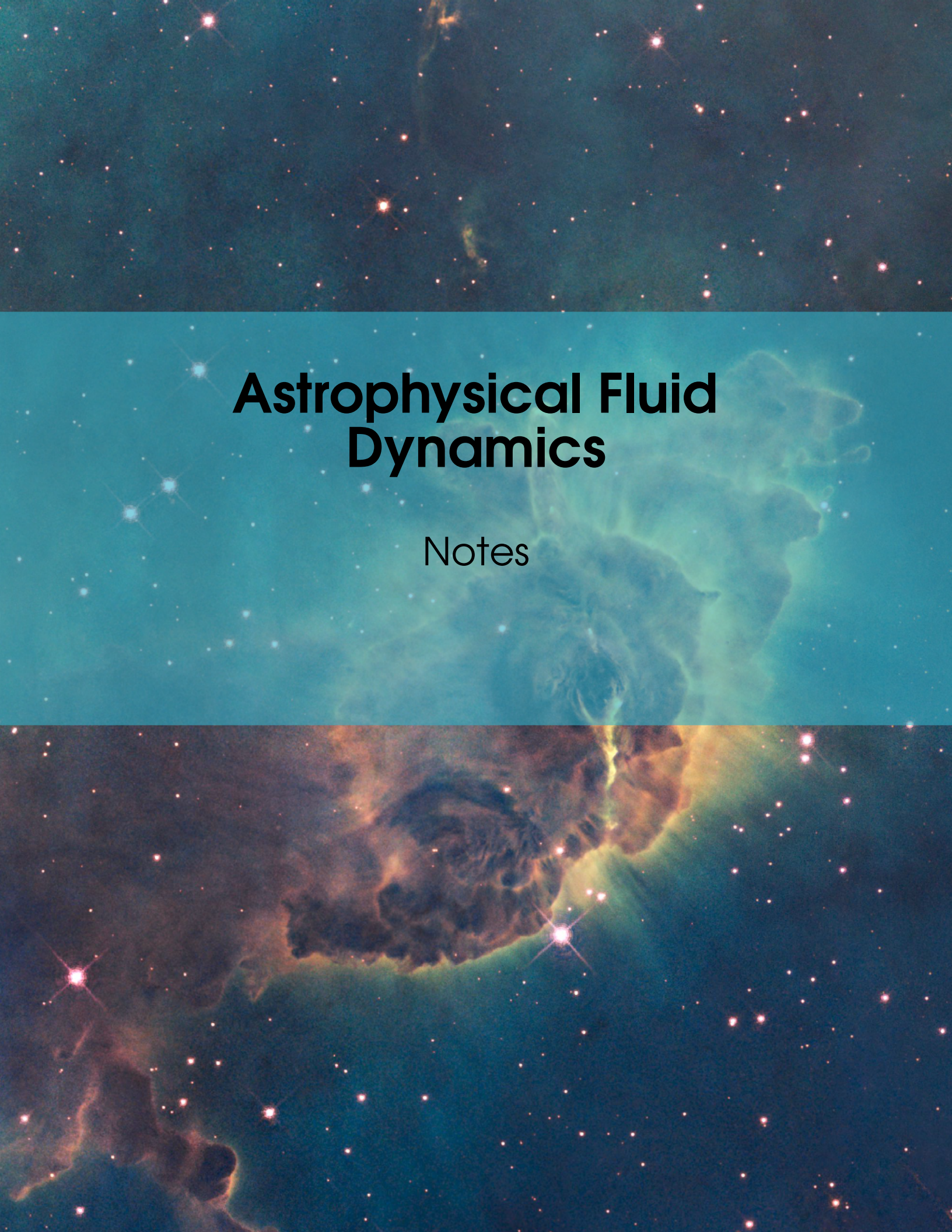

Astrophysical Fluid Dynamics

Notes

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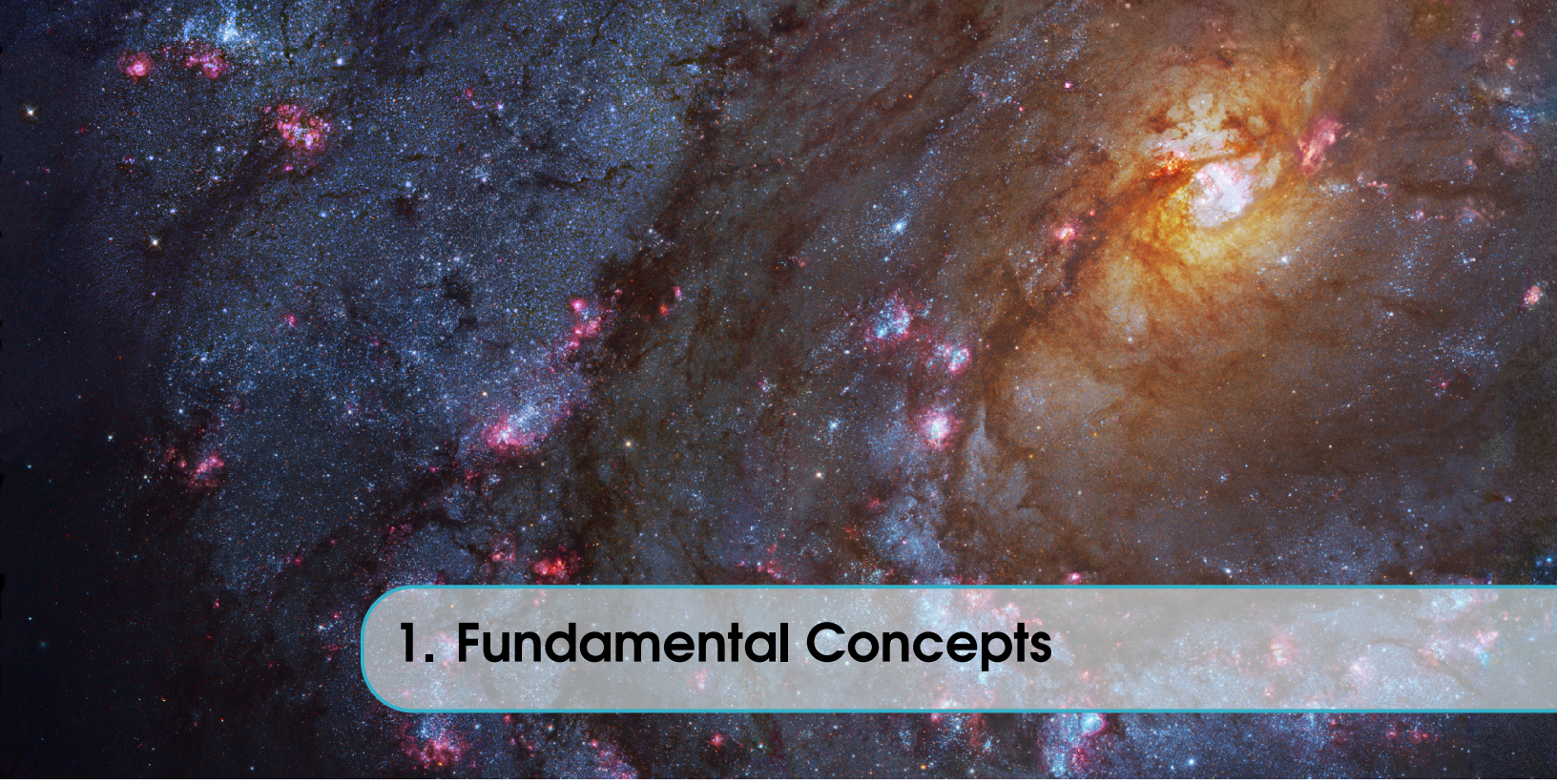
Contents

Part I

Fundamentals of Fluid Dynamics



1. Fundamental Concepts

As it concerns our purposes in these notes, a fluid may be (roughly) defined as

Definition 1.0.1 — Fluid. A fluid is some spatially distributed material which is able to freely deform. In effect, this distinguishes solids from fluids (gas and liquid).

R It isn't hard to find instances where this sort of treatment breaks down; however, in astrophysical scenarios we are almost always concerned with material which is distinctively gaseous and therefore do not care to make too much effort in refining the above definition.

In classical fluid dynamics, the **incompressibility** of the flow is often introduced early and kept as a core assumption throughout. This is not a valid approximation in most astrophysical scenarios as gas is often compressed. We therefore do not benefit from any of the theory regarding incompressible flow.

1.1 Elementary Definitions

To begin our discussion, we will make some elementary remarks about important definitions and concepts.

1.1.1 The Fluid Element

Fluid Dynamics is, first and foremost, an **effective (ensemble) theory**, meaning it describes the collective behavior of an enormous number of discrete constituents (particles) using a small set of continuous fields, such as the mass density ρ and velocity field $\mathbf{u}$. For this coarse-grained description to be valid, we must identify a characteristic length scale ℓ satisfying two key conditions:

1. **Sufficiently small:** The scale ℓ must be much smaller than the length scales over which any relevant quantity q varies appreciably. In other words, relative variations

$\delta q/q$ across ℓ should satisfy

$$\delta q/q \ll 1 \tag{1.1}$$

This ensures we can meaningfully associate a well-defined, approximately uniform value of q to each fluid element, neglecting internal fluctuations within that element.

In addition,

2. **Sufficiently statistical:** The scale ℓ must be large enough to contain many particles, so that microscopic, particle-level fluctuations are negligible compared to the collective, ensemble behavior. Formally, this requires the particle number density n to satisfy

$$\ell^3 n \gg 1. \tag{1.2}$$

The above criteria are sufficient to define a **fluid** in the context of an **effective field theory**, where the microscopic particle description is replaced by continuous fields. However, even when these conditions are met, a subtle distinction remains concerning the dynamical behavior of the system.

In general, such a fluid may retain some degree of **phase-space memory**: its present evolution can depend not only on the instantaneous values of the fluid fields, but also on the detailed history of the system's distribution function. In other words, perturbations to the microscopic particle distribution at an earlier time δt may still influence the system's current state.

The extent to which this phase-space memory affects the dynamics depends on whether the fluid is **collisional** or **collisionless**:

- In a **collisional fluid**, frequent particle interactions drive the system toward local thermodynamic equilibrium on timescales short compared to the macroscopic evolution. In this regime, the fluid fields ρ, $\mathbf{u}$, etc., contain all necessary information, and the system rapidly forgets its microscopic history. The dynamics are thus determined entirely by local field values and their gradients. In order for a fluid to be **collisional**, we require that the **mean free path** between interactions satisfy

$$\lambda_{\text{mfp}} \ll \ell. \tag{1.3}$$

Recalling that

$$\lambda_{\text{mfp}} = \frac{1}{n\sigma} \implies n\sigma \gg \ell^{-1}. \tag{1.4}$$

- In a **collisionless fluid**, particle interactions are sufficiently rare or absent that the system does not locally equilibrate. Instead, the evolution retains dependence on the detailed particle distribution function in phase space. In this case, a purely field-based description is incomplete, and additional information about the microscopic stateoften encapsulated by the distribution function $f(\mathbf{x}, \mathbf{p}, t)$is required.

Thus, while the field-theoretic description of fluids is powerful, its completeness depends critically on whether the underlying system rapidly equilibrates (collisional) or preserves phase-space structure over dynamical timescales (collisionless).

1.1.2 View of Fluid Dynamics: Eulerian and Lagrangian Perspectives

Fluid dynamics can be formulated from two mathematically distinct but physically equivalent viewpoints: the **Eulerian** and the **Lagrangian** descriptions. These perspectives differ in how they **label fluid elements** and in what domain the physical fields are defined.

Consider a continuum which, at initial time $t = 0$, occupies a reference configuration $\mathcal{C}_0 \subset \mathbb{R}^3$. As time progresses, the fluid deforms into new spatial configurations $\mathcal{C}_t$, flowing and evolving through space. This notion leads us to the rather formal definition of the ***flow map****, which tells us how to go from one deformation space to a later one.*

Definition 1.1.1 — Flow Map / Configuration Map. Let $\mathcal{C}_0 \subset \mathbb{R}^3$ be the reference configuration describing the fluid position at $t = t_0$. The **flow map**

$$\boldsymbol{\varphi} : \mathcal{C}_0 \times \mathbb{R} \to \mathbb{R}^3$$

is defined by

$$\boldsymbol{\varphi}(\mathbf{X}, t) = \mathbf{x},$$

where $\mathbf{X} \in \mathcal{C}_0$ is the **label** of a material particle (its initial position), and $\boldsymbol{\varphi}(\mathbf{X}, t)$ gives its position in physical space at time t. For each fixed t, the map $\boldsymbol{\varphi}_t := \boldsymbol{\varphi}(\cdot, t)$ takes the reference configuration to the **current configuration** $\mathcal{C}_t = \boldsymbol{\varphi}_t(\mathcal{C}_0) \subset \mathbb{R}^3$.

This flow map naturally defines the velocity field:

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\varphi}}{\partial t}(\mathbf{X}, t)\bigg|_{\mathbf{X}=\boldsymbol{\varphi}^{-1}(\mathbf{x},t)}.$$

The unfortunate outcome of this picture is that there are **two equally good ways to look at the world**. We could label every particle based on $\mathcal{C}_0$ and then use φ to move forward into C_t, or we could label particles based on C_t and go backward with φ^{-1}. These are both perfectly good ways to look at the world and they are beneficial in different scenarios. Formally,

Definition 1.1.2 — Eulerian Field. In the **Eulerian** description, fluid properties are described as functions on the **current configuration** $\mathcal{C}_t \subset \mathbb{R}^3$. That is, a scalar field ψ (such as pressure, density, etc.) is defined by:

$$\psi : \mathcal{C}_t \to \mathbb{R}, \quad \mathbf{x} \mapsto \psi(\mathbf{x}, t),$$

where $\mathbf{x} \in \mathcal{C}_t$ is a point in space occupied by the fluid at time t, and $\psi(\mathbf{x}, t)$ gives the value of the field at that location.

If we wish to express this quantity in terms of the **reference configuration** $\mathcal{C}_0$, we must **pull back** the field along the flow map $\boldsymbol{\varphi} : \mathcal{C}_0 \times \mathbb{R} \to \mathcal{C}_t$. The associated Lagrangian

field is:

$$\Psi(\mathbf{X}, t) := \psi(\boldsymbol{\varphi}(\mathbf{X}, t), t),$$

where $\mathbf{X} \in \mathcal{C}_0$ labels a material particle.

Conversely, the Eulerian field is the **pushforward** of the **Lagrangian field**:

$$\psi(\mathbf{x}, t) = \Psi(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t), \quad \text{for } \mathbf{x} \in \mathcal{C}_t.$$

As was hinted about in definition 1.1.2, the **Lagrangian view** is effectively the inverse:

Definition 1.1.3 — Lagrangian Field. In the **Lagrangian** description, fluid properties are described as functions on the **reference configuration** $\mathcal{C}_0 \subset \mathbb{R}^3$, which labels material particles by their initial positions.

Let $\boldsymbol{\varphi} : \mathcal{C}_0 \times \mathbb{R} \to \mathbb{R}^3$ be the **flow map**, such that for each material label $\mathbf{X} \in \mathcal{C}_0$, $\boldsymbol{\varphi}(\mathbf{X}, t)$ gives the spatial position of that particle at time t.

Then, for a physical observable $\psi : \mathcal{C}_t \to \mathbb{R}$, the corresponding **Lagrangian field** is defined by the **pullback** of ψ along $\boldsymbol{\varphi}$:

$$\Psi : \mathcal{C}_0 \times \mathbb{R} \to \mathbb{R}, \qquad \Psi(\mathbf{X}, t) := \psi(\boldsymbol{\varphi}(\mathbf{X}, t), t).$$

That is, $\Psi(\mathbf{X}, t)$ gives the value of the field as experienced by the material particle labeled by $\mathbf{X}$, as it evolves in time.

Conversely, the Eulerian field can be recovered by the **pushforward**:

$$\psi(\mathbf{x}, t) = \Psi(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t), \quad \text{for } \mathbf{x} \in \mathcal{C}_t.$$

R In general, **Eulerian** methods are more powerful for most uses analytically. We are generally more interested in the benefits of having a simple, stationary coordinate system than the benefits of being able to track individual packets of material. There are some exceptions to this however.

In computational fluid dynamics, this is dramatically different as both approaches lead to powerful computational methods.

We can naturally connect the **Eulerian** and **Lagrangian** descriptions of fluid dynamics. Consider a flow characterized by a **velocity field** $\mathbf{u}(\mathbf{r}, t)$ and a scalar field $\psi(\mathbf{r}, t)$ defined in the Eulerian frame.

Suppose a fluid particle is initially located at position $\mathbf{r}_0$ at time $t = 0$. After an infinitesimal time interval δt, the particle moves to

$$\mathbf{r}_1 = \mathbf{r}_0 + \mathbf{u}(\mathbf{r}_0, 0)\, \delta t.$$

The change in ψ experienced by the particle along its trajectorythe **Lagrangian change**is given by

$$\delta\psi_{\text{lagrangian}} = \psi(\mathbf{r}_1, \delta t) - \psi(\mathbf{r}_0, 0).$$

Assuming ψ is sufficiently smooth (at least C^2 continuous), we expand ψ to first order:

$$\psi(\mathbf{r}_1, \delta t) \approx \psi(\mathbf{r}_0, 0) + \delta\mathbf{r} \cdot \nabla\psi + \delta t \, \frac{\partial \psi}{\partial t} + \mathcal{O}(\delta\mathbf{r}^2, \delta t^2).$$

Since $\delta\mathbf{r} = \mathbf{u}\,\delta t$, this becomes:

$$\delta\psi_{\text{lagrangian}} = \mathbf{u} \cdot \nabla\psi \, \delta t + \frac{\partial \psi}{\partial t} \, \delta t + \mathcal{O}(\delta t^2).$$

Dividing both sides by δt and taking the limit $\delta t \to 0$, we obtain the **material derivative** (or **Lagrangian derivative**):

$$\boxed{\frac{d\psi_{\text{lagrangian}}}{dt} = \frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla\psi.} \tag{1.5}$$

This expression shows how the time rate of change of a field ψ as experienced by a moving fluid element (Lagrangian perspective) relates to the local, fixed-position time derivative and spatial variations in the Eulerian description.

R It is important to remember that the **Lagrangian derivative** is NOT written in the lagrangian frame. Really, we are representing D/Dt in the Eulerian frame as $\partial_t + \mathbf{u}\cdot\nabla$, which then reminds us that, formally, we regard $\mathbf{u}$ as $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\varphi(\mathbf{X}, t), t)$ and ∇ as $\nabla_{\mathbf{x}}$, not $\nabla_{\mathbf{X}}$.

1.2 Structure of the Velocity Field

An extremely important field present in fluid dynamical flows is the **velocity field u**, which describes the instantaneous velocity at a point $(x, t) \in \mathbb{R}^d \times \mathbb{R}$. There are a number of things to be said about this velocity field which are relevant throughout this subject.

1.2.1 The Velocity Gradient

A common construct in fluid kinematics is the **velocity gradient**, which is a $(1, 1)$ tensor field over the domain defined such that

$$\mathbf{V}(\omega, X) = \omega(X(\mathbf{u}))\,.$$

In a particular basis, this is equivalent to

$$V^\nu_\mu = \nabla_\mu u^\nu,$$

where ∇_μ is the **covariant derivative**. *In Cartesian coordinates, this may be taken as the standard gradient.*

Like any rank-2 tensor, the velocity gradient can be decomposed into its symmetric and antisymmetric parts:

$$V^\nu_\mu = \tfrac{1}{2}\left(\nabla_\mu u^\nu + \nabla_\nu u^\mu\right) + \tfrac{1}{2}\left(\nabla_\mu u^\nu - \nabla_\nu u^\mu\right) = S^\nu_\mu + \Omega^\nu_\mu.$$

The Rate of Strain Tensor

The symmetric component is called the **rate-of-strain tensor**:

Definition 1.2.1 — Rate-of-Strain Tensor. The **rate-of-strain tensor** (or **strain rate tensor**) is defined by

$$S_\mu^\nu = \tfrac{1}{2}\left(\nabla_\mu u^\nu + \nabla_\nu u^\mu\right).$$

It encodes both the isotropic *expansion* of fluid elements and the *shear* which distorts their shape without changing volume.

The trace of $S_{\mu\nu}$ yields the **expansion scalar**:

$$\theta = \nabla_\mu u^\mu,$$

which measures the local volumetric dilation of the flow. Subtracting this isotropic part leaves the **shear tensor**:

$$\sigma_{\mu\nu} = S_{\mu\nu} - \tfrac{1}{3}\theta\, g_{\mu\nu},$$

a symmetric, trace-free object describing pure shape distortion.

This decomposition is directly relevant for fluid dynamics: in Newtonian fluids the viscous stress tensor is proportional to $S_{\mu\nu}$, with separate coefficients (bulk and shear viscosity) multiplying its trace and trace-free parts. Thus $S_{\mu\nu}$ is the quantity that determines how velocity gradients are converted into internal stresses and, ultimately, into heat through viscous dissipation.

The Vorticity Tensor

The antisymmetric component is called the **vorticity tensor**:

Definition 1.2.2 — Vorticity Tensor. The **vorticity tensor** is defined by

$$\Omega_\mu{}^\nu = \tfrac{1}{2}\left(\nabla_\mu u^\nu - \nabla_\nu u^\mu\right).$$

It encodes the local *rotation* of fluid elements, i.e. the rigid-body spin of a small fluid parcel.

In three-dimensional Euclidean space, $\Omega_{\mu\nu}$ corresponds directly to the familiar **vorticity vector**

$$\boldsymbol{\omega} = \nabla \times \mathbf{u},$$

which provides an intuitive picture of the local axis of rotation of the flow.

Vorticity is central in the classification of flows. When $\Omega_{\mu\nu} = 0$, the flow is called **irrotational**, and the velocity can be expressed as the gradient of a scalar potential. This condition underlies many classical results such as Bernoullis theorem. Conversely, nonzero vorticity characterizes rotational flows, vortex dynamics, and ultimately turbulence, where the stretching and diffusion of vorticity dominate the behavior of the system.

Summary

Altogether, the velocity gradient admits a canonical decomposition:

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \tfrac{1}{3}\theta\, g_{\mu\nu} + \Omega_{\mu\nu},$$

into *shear*, *expansion*, and *vorticity*. Each piece has distinct physical significance:

- θ controls local compression or expansion of the fluid.
- $\sigma_{\mu\nu}$ governs shape distortion and is directly linked to viscous stresses and dissipation.
- $\Omega_{\mu\nu}$ (or $\boldsymbol{\omega}$) measures local rigid-body rotation, with vanishing vorticity defining irrotational flow.

This decomposition provides a complete kinematic description of how fluid elements evolve in time.

Big Idea

The **velocity gradient** $\nabla_\mu u_\nu$ can always be decomposed into three geometrically and physically distinct parts:

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \tfrac{1}{3}\theta\, g_{\mu\nu} + \Omega_{\mu\nu}.$$

This canonical splitting encodes the full local behavior of a fluid element:

- θ: the **expansion scalar**, measuring isotropic compression or dilation (volume change).
- $\sigma_{\mu\nu}$: the **shear tensor**, capturing pure shape distortion. It is the driver of viscous stresses and dissipation.
- $\Omega_{\mu\nu}$: the **vorticity tensor**, representing rigid-body rotation of the fluid element. In 3D, this is equivalent to the familiar vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, whose vanishing defines *irrotational flow*.

Together, these components form the cornerstone of fluid kinematics: they describe how small parcels of fluid move, deform, and spin, and they connect directly to the physics of viscosity, potential flows, and turbulence.

2. The Fluid Equations

In this section, we will derive the critical equations of fluid dynamics: the so-called **Euler Equations**. In effect, these are derived from 3 core principles:

1. **Conservation of Mass**: leads to the *conservation equation*.
2. **Conservation of Energy / Momentum**: lead to the Euler equations.

2.1 The Continuity Equation

Consider a fluid with a conserved density field ψ (generally the **mass density**). In some region V, the total amount of the conserved field is

$$\Psi = \int_V \psi dV.$$

Now, this may flow in / out of the volume or be created / destroyed within the volume. In some time δt, we describe the created / destroyed quantity as

$$\Delta\Psi_{\text{internal}} = \int_V \delta\psi dV$$

Additionally, there is some flow in / out of the surface

$$\Delta\Psi_{\text{flux}} = dt \oint_{\partial V} \psi\mathbf{u} \cdot d\mathbf{S}.$$

If **no material is created / destroyed**, then

$$\frac{d\Psi}{dt} = -\oint_{\partial V} \psi\ \mathbf{u} \cdot d\mathbf{S}$$

By **divergence theorem**,

$$-\oint_{\partial V} \mathbf{X} \cdot d\mathbf{S} = -\int_V \nabla \cdot \mathbf{X} dV \implies \oint_{\partial V} \psi\mathbf{u} \cdot d\mathbf{S} = \int_V \nabla \cdot (\psi\mathbf{u}) dV.$$

Therefore,

$$\frac{d\Psi}{dt} = \int_V \frac{\partial \psi}{\partial t} dV = -\int_V \nabla \cdot (\psi \mathbf{u}) dV.$$

In the limit as $V \to 0$, this becomes

$$\boxed{\frac{\partial \psi}{\partial t} + \nabla \cdot (\psi \mathbf{u}) = 0. \text{ Eulerian.}} \tag{2.1}$$

R In most cases, we are chiefly concerned with the density ρ, in which case we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

In the frequent case (in classical fluid dynamics) where ρ is a constant of the flow, you can pull it out; however, this is not generally the case in compressible gas flows.

To write the above result in **Lagrangian form**, we need only remember that (equation 1.5)

$$\frac{\partial \psi_{\text{Eulerian}}}{\partial t} = \frac{D\psi}{Dt} - \mathbf{u} \cdot \nabla \psi.$$

Recall that $\nabla \cdot (\rho \mathbf{u}) = \partial_k (\rho u^k) = \rho \partial_k u^k + u^k \partial_k \rho = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho$. Thus, substituting into equation 2.1, we find

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \text{ Lagrangian.}} \tag{2.2}$$

R The intuition here is not as easy to see on inspection. Imagine yourself floating along in the flow. The first term $D\rho/Dt$ describes how the **comoving density** behaves (the density you feel). Now, what should that intuitively be tied to? Well clearly to the density of the particles around you. They are all also moving with the flow, so as long as the flow lines remain totally parallel we don't get any change in density. If the flow lines converge, then the density will clearly go up as particles get pressed closer. The natural measure of that "convergence" is the **divergence of the velocity field**.

Notice that both forms of the equation have their "problem terms." The Eulerian approach requires $\nabla \cdot (\psi \mathbf{u})$, which may be tricky to evaluate. Meanwhile, the Lagrangian term only requires $\nabla \cdot \mathbf{u}$ but also requires $D\rho/Dt$.

R Consider the scenario where $D\rho/Dt = 0$. We see immediately that this requires $\nabla \cdot \mathbf{u} = 0$, so the fluid is **solenoidal** (divergenceless). This is a characteristic aspect of **divergence free flows** or **incompressible flows**, which occur in many limits.

2.2 Vector Conservation Laws

Just as we did for the scalar field ψ, we might also be interested in the conservation of a vector density field $\mathbf{\Psi}$ defined on the domain. This is a **trickier calculation**; however, it has a massive role in the nature of fluid dynamics, especially in the conservation of momentum.

> R For a **vector density**, each component of the field must be conserved as in the scalar case; however, flux in / out of the surface in a particular index i may couple with the j component in the adjacent region. This was a consideration which was not necessary in the treatment of the scalar case.

Consider a region V of the domain with total quantity of ψ_k:

$$\Psi_k = \int_V \psi_k \, dV.$$

As in the scalar case, changes to Ψ_k may arise from creation/annihilation within V and flux across the boundary ∂V. The creation/annihilation is straightforward:

$$\Delta\Psi_{k,\text{ internal}} = \int_V \frac{\partial \psi_k}{\partial t} \, dV.$$

However, the movement across the membrane is **not as simple to describe**, since the vector field may not transport identically across all orientations of the membrane, nor in all components. We therefore introduce the **Flux Tensor** Σ^k_j, defined such that:

$$\Delta\Psi_{k,\text{ flux}} = \oint_{\partial V} \Sigma^k_j \, dS^j.$$

Note that we have made a sign convention here which needs to be kept track of... Here, dS^j is the oriented surface area element in the j^{th} direction, and Σ^k_j expresses the flux of the k^{th} component of the conserved vector through surfaces with orientation in the j^{th} direction. The change in the total Ψ in a time Δt is thus

$$\Delta\Psi = \int_V \frac{\partial \psi_k}{\partial t} \, dV + \oint \Sigma^k_j \, dS^j = \int_V \frac{\partial \psi_k}{\partial t} + \partial_j \Sigma^k_j \, dV \Delta t$$

Now, the LHS provides source terms to the equation and we therefore have that, for some set of sources $\mathbf{f}$,

$$\frac{\partial \boldsymbol{\psi}}{\partial t} + \nabla \cdot \mathbf{\Sigma} = \mathbf{f}.$$

Since V is arbitrary, the integrands must be equal everywhere, leading to the **general vector conservation law**:

$$\boxed{\frac{\partial \psi_k}{\partial t} + \partial_j \Sigma^k_j = 0.} \tag{2.3}$$

In **vector notation**, this is

$$\frac{\partial \mathbf{\Psi}}{\partial t} + \nabla \cdot \mathbf{\Sigma} = 0, \text{ Eulerian.}$$

2.2.1 Momentum Conservation

With the general form of a **vector conservation law** in hand, we now want to derive the explicit form of the flux tensor Σ for the case of momentum conservation. The relevant vector density field is the **momentum density**:

$$\mathbf{p} = \rho \mathbf{u},$$

where ρ is the mass density and $\mathbf{u}$ is the velocity field.

There are **two distinct physical mechanisms** by which momentum is transported in a fluid:

1. **Advection**: Material crosses the boundary ∂V carrying its momentum with it. The mass flux across a surface with oriented area element dS^j is $\rho u_j \, dS^j$. Since each unit of mass carries momentum $\mathbf{p}$, the k^{th} component of momentum advected across the surface is:

$$\Delta p^k_{\text{advective}} = \rho u^k u_j \, dS^j.$$

2. **Internal Stresses**: Even without bulk motion, microscopic interactions between particles on either side of the boundary may transfer momentum. These are summarized by the **stress tensor** σ^k_j, defined such that the k^{th} component of momentum transferred per unit area per unit time across a surface with normal $\hat{n}^j$ is:

$$\Delta p^k_{\text{stress}} = \sigma^k_j \, \hat{n}^j.$$

Combining these mechanisms, the total momentum flux tensor takes the form:

$$\Sigma^k_j = \rho u^k u_j + \sigma^k_j.$$

The general vector conservation law for momentum density reads:

$$\frac{\partial p_k}{\partial t} + \partial_j \Sigma^k_j = 0,$$

or explicitly:

$$\frac{\partial (\rho u_k)}{\partial t} + \partial_j \left(\rho u_k u^j + \sigma^k_j \right) = 0.$$

In vector notation, this becomes:

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \otimes \mathbf{u}) + \nabla \cdot \boldsymbol{\sigma} = 0,$$

where:

- $\mathbf{p} \otimes \mathbf{u}$ denotes the outer product (dyadic product) of momentum and velocity, producing a rank-2 tensor with components $p^k u_j = \rho u^k u_j$,
- $\boldsymbol{\sigma}$ is the stress tensor.

This formulation clearly separates:

- Momentum transport due to advection ($\mathbf{p} \otimes \mathbf{u}$),
- Momentum exchange due to internal stresses ($\boldsymbol{\sigma}$).

In **some cases**, we may have **external sources of momentum**, leading us to

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \otimes \mathbf{u}) + \nabla \cdot \boldsymbol{\sigma} = \mathbf{F}_{\text{ext}}.$$

R We also might characterize the entire RHS as a stress tensor:

$$\boldsymbol{\sigma} = (\mathbf{p} \otimes \mathbf{u} + P\mathbf{I}),$$

in which case, we have

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot \boldsymbol{\sigma} = \mathbf{F}_{\text{ext}}.$$

much more can be done with a bit more work.

2.2.2 The Stress Tensor

We have, so far, derived the critical equation

$$\boxed{\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \otimes \mathbf{u}) + \nabla \cdot \boldsymbol{\sigma} = \mathbf{F}_{\text{ext}}.}$$

for a very general $\boldsymbol{\sigma}$, these equations can be quite difficult to solve; however, we can make quite a bit of progress on physical arguments about the stress tensor. In this section, we'll take some time to explore these behaviors before proceeding.

Theorem 2.2.1 — The stress tensor is symmetric. Let $\mathbf{p}$ be the momentum of a particular fluid flow with velocity field $\mathbf{u}$ and density field ρ. Regardless of the nature of the system, the **stress tensor** σ_{ij} is everywhere symmetric.

*In fact, by the same logic, you can show that ANY **flux tensor** in a momentum conservation law MUST have these properties for the same reason. This makes the flux tensor for momentum special in that it must ALWAYS be symmetric.*

Proof. The intuition behind this is relatively simple. Consider a small control volume of fluid with negligible size. If the off-diagonal elements of the stress tensor are not symmetric, the internal forces will generate a net torque proportional to the surface area of the control volume. As the size of the volume shrinks to zero, the torque remains finite, implying the possibility of unbounded angular acceleration contradicting the assumption of local mechanical equilibrium.

We now present the formal derivation based on the conservation of angular momentum. The total angular momentum of a region V is given by:

$$L^k = \int_V \epsilon^{kij} x_i \rho u_j \, dV,$$

where:

- ϵ^{kij} is the Levi-Civita symbol,
- x_i is the position vector,

- ρu_j is the j^{th} component of the momentum density.

The rate of change of angular momentum is:

$$\frac{dL^k}{dt} = \int_V \epsilon^{kij} x_i \frac{\partial(\rho u_j)}{\partial t}\, dV.$$

Applying the momentum conservation equation:

$$\frac{\partial(\rho u_j)}{\partial t} + \partial_m \Sigma_j^m = 0,$$

we substitute:

$$\frac{dL^k}{dt} = -\int_V \epsilon^{kij} x_i \partial_m \Sigma_j^m\, dV.$$

Using the product rule and the divergence theorem:

$$\frac{dL^k}{dt} = -\oint_{\partial V} \epsilon^{kij} x_i \Sigma_j^m\, dS_m + \int_V \epsilon^{kij} \Sigma_j^m \delta_{im}\, dV.$$

The first term represents the torque exerted on the boundary of the region, which may be nonzero due to external forces. The second term simplifies using δ_{im}:

$$\frac{dL^k}{dt} = -\oint_{\partial V} \epsilon^{kij} x_i \Sigma_j^m\, dS_m + \int_V \epsilon^{kij} \Sigma_j^i\, dV.$$

In the absence of internal microscopic torques (body couples), conservation of angular momentum requires that the internal torque contribution vanishes identically:

$$\int_V \epsilon^{kij} \Sigma_j^i\, dV = 0.$$

Since V is arbitrary, the integrand itself must vanish pointwise:

$$\epsilon^{kij} \Sigma_j^i = 0.$$

The Levi-Civita symbol is antisymmetric in i and j, implying that the antisymmetric part of Σ_j^i must vanish. Explicitly, we can decompose:

$$\Sigma_j^i = \Sigma^{(ij)} + \Sigma^{[ij]},$$

where:

$$\Sigma^{(ij)} = \frac{1}{2}\left(\Sigma_j^i + \Sigma_i^j\right), \quad \Sigma^{[ij]} = \frac{1}{2}\left(\Sigma_j^i - \Sigma_i^j\right).$$

The expression $\epsilon^{kij}\Sigma_j^i = 0$ requires the antisymmetric part $\Sigma^{[ij]}$ to vanish identically. Therefore, the stress tensor is symmetric:

$$\boxed{\Sigma_j^i = \Sigma_i^j}.$$

In particular, for the internal stress tensor σ_j^i, this symmetry property holds:

$$\sigma_j^i = \sigma_i^j.$$

Thus, the stress tensor is symmetric everywhere, consistent with conservation of angular momentum in the absence of internal body torques.

■

Theorem 2.2.1 gives us a few very interesting corollaries:

Corollary 2.2.2 For any stress tensor σ_{ij}, there is a coordinate frame (orthogonal transformation) in which σ is **diagonalized**. In this frame, there are no shear forces and the expansion / contraction of fluid elements is more directly understood.

Corollary 2.2.3 Any stress tensor may be **decomposed** into its **isotropic** and **deviatoric** components:

- The **isotropic component** is of the form $-p\delta_{ij}$, where p is the **very familiar quantity of pressure**.
- The **deviatoric component** is effectively everything else: τ_{ij}. It describes the shears and viscous effects.

Boundary Conditions

In continuum mechanics, boundary conditions arise from the requirement that forces on surfaces balance. The relevant object is the Cauchy stress tensor $\boldsymbol{\sigma}$, which relates to the traction vector (force per unit area) acting on a surface with unit normal $\hat{\mathbf{n}}$ via

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}. \tag{2.4}$$

At an interface between two media, **Newtons third law requires** that the tractions exerted on the surface by each medium, together with any intrinsic surface forces, must balance. In the most general form:

$$\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \cdot \hat{\mathbf{n}} = \nabla_s \cdot \boldsymbol{\tau}_s, \tag{2.5}$$

where

- $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ are the bulk stress tensors on either side of the interface,
- $\hat{\mathbf{n}}$ is the unit normal to the interface (pointing into medium 1),
- $\boldsymbol{\tau}_s$ is the surface stress tensor (e.g. isotropic surface tension),
- ∇_s denotes the surface divergence operator.

This condition can be decomposed into two parts:

1. **Normal stress balance:**

$$\hat{\mathbf{n}} \cdot \left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \left(\nabla_s \cdot \boldsymbol{\tau}_s\right), \tag{2.6}$$

   which determines the pressure jump across the surface (including, e.g., Laplace pressure if surface tension is present).

2. **Tangential stress continuity:**

$$\hat{\mathbf{t}} \cdot \left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \cdot \hat{\mathbf{n}} = 0, \tag{2.7}$$

   for all tangent directions $\hat{\mathbf{t}}$, unless additional surface shear stresses are present.

Physical interpretation. The boundary condition is simply the statement that the interface, having no bulk inertia, cannot sustain a net unbalanced traction. The normal component enforces pressure or normal stress balance, while the tangential component enforces shear stress continuity. Any intrinsic surface physics (e.g. isotropic tension, membrane elasticity) enters through the surface stress tensor $\boldsymbol{\tau}_s$.

2.3 Euler's Equations

We are now in a position to derive the **Euler equations** the fundamental equations governing the motion of an *inviscid* (non-viscous) fluid. These equations follow naturally by simplifying the general momentum conservation law under the assumption that viscous stresses vanish and only isotropic pressure remains.

Momentum Conservation in General Form

Recall the momentum conservation equation:

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \otimes \mathbf{u}) + \nabla \cdot \boldsymbol{\sigma} = \mathbf{F}_{\text{ext}},$$

where:

- $\mathbf{p} = \rho \mathbf{u}$ is the momentum density,
- $\boldsymbol{\sigma}$ is the internal stress tensor,
- $\mathbf{F}_{\text{ext}}$ represents external body forces (e.g., gravity).

For an inviscid fluid, there are no shear or viscous stresses. The only internal stress is the isotropic pressure, so:

$$\sigma_{ij} = -p\delta_{ij},$$

where:

- p is the pressure,
- δ_{ij} is the Kronecker delta.

Substituting into the momentum conservation equation:

$$\frac{\partial(\rho u_k)}{\partial t} + \partial_j \left(\rho u_k u^j\right) + \partial_j \left(-p\delta_j^k\right) = F_{\text{ext}}^k.$$

Simplifying:

$$\frac{\partial(\rho u_k)}{\partial t} + \partial_j \left(\rho u_k u^j\right) = -\partial_k p + F_{\text{ext}}^k.$$

Expanding with the Continuity Equation

We can rewrite the time derivative term using the product rule:

$$\frac{\partial(\rho u_k)}{\partial t} = \rho \frac{\partial u_k}{\partial t} + u_k \frac{\partial \rho}{\partial t},$$

and similarly expand the divergence term:

$$\partial_j \left(\rho u_k u^j\right) = \rho u^j \partial_j u_k + u_k \partial_j (\rho u^j).$$

Thus, the momentum equation becomes:

$$\rho \frac{\partial u_k}{\partial t} + u_k \frac{\partial \rho}{\partial t} + \rho u^j \partial_j u_k + u_k \partial_j (\rho u^j) = -\partial_k p + F_{\text{ext}}^k.$$

We now use the continuity equation (conservation of mass):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

or explicitly:

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u^j) = 0.$$

Substituting this into the momentum equation, notice:

$$u_k \left(\frac{\partial \rho}{\partial t} + \partial_j(\rho u^j) \right) = 0.$$

Therefore, the momentum equation simplifies to:

$$\rho \left(\frac{\partial u_k}{\partial t} + u^j \partial_j u_k \right) = -\partial_k p + F^k_{\text{ext}}.$$

Lagrangian Form of the Euler Equations

The left-hand side of the above is the **material derivative** of u_k:

$$\frac{Du_k}{Dt} = \frac{\partial u_k}{\partial t} + u^j \partial_j u_k.$$

Thus, the **Lagrangian form** of the Euler equations is:

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F}_{\text{ext}}.} \tag{2.8}$$

Intuitively, this is the co-moving conservation of momentum equation, which makes a lot of sense on examination of its various terms and in analogy with the familiar form of Newton's 2nd law.

Eulerian Form of the Euler Equations

Alternatively, from the stationary (Eulerian) perspective, we can simply transform the equation as

$$\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{F}_{\text{ext}}.$$

This is equivalent to

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \otimes \mathbf{u}) + \nabla \cdot \boldsymbol{\sigma} = \mathbf{F}_{\text{ext}},$$

after reduction of the stress tensor and inclusion of the conservation of mass.

In the Lagrangian viewpoint, you imagine yourself drifting along with a fluid parcel. You directly experience how the velocity of the parcel changes over time as a result of local pressure gradients and external forces, such as gravity. The material derivative $D\mathbf{u}/Dt$ captures this physical acceleration of the parcel.

Summary of the Inviscid Euler Equations

For an inviscid, compressible fluid with mass density ρ, velocity field $\mathbf{u}$, and pressure p, the governing equations are:

1. **Conservation of Mass (Continuity Equation)**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

2. **Conservation of Momentum (Euler Equations)**:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F}_{\text{ext}}.$$

These two equations form the foundation of inviscid fluid dynamics. They are generally closed by an **equation of state** relating pressure p to ρ and potentially other thermodynamic quantities, especially in compressible flows.

2.4 Viscosity

So far in our discussion of momentum conservation, we have discussed only scenarios with **bulk transfer of momentum**:

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \otimes \mathbf{u}) + \nabla \cdot \boldsymbol{\sigma} = \mathbf{F}_{\text{ext}},$$

where $\boldsymbol{\sigma}$ is an **isotropic stress tensor** containing only the contribution of thermal pressure. In this section, we'll discuss the advent of **viscosity** and its role in the dynamics of fluids.

2.4.1 Intuition for Viscosity

Viscosity originates from the **microscopic transport of momentum by particles in thermal motion.** Even in a simple gas, molecules do not remain confined to a single "layer" of the fluid: they constantly move across notional boundaries separating adjacent regions. When a molecule crosses such a boundary, it carries with it the momentum characteristic of its layer of origin. If the bulk velocity differs between neighboring layers, these molecular exchanges create a net flux of momentum.

This picture explains why **velocity gradients** are the key ingredient in viscous phenomena. When the flow is spatially uniform, particle exchange across layers simply mixes identical momentum distributions, producing no net effect. By contrast, if there is a velocity gradient, the molecules arriving from the faster layer carry more momentum than those arriving from the slower layer. The result is an imbalance: momentum is transported down the gradient, and the fluid experiences a **shear stress** opposing the relative motion of the layers.

A simple kinetic estimate makes this idea quantitative. Consider a shear flow with velocity $u(y)$ in the x–direction. Molecules originating a distance of order the mean free path ℓ away bring with them an excess (or deficit) of momentum proportional to $\rho\, v_{\text{th}}\, \ell\, \partial u/\partial y$, where ρ is the mass density and v_{th} the typical thermal speed. This leads to an effective shear force per unit area

$$\tau_{xy} \sim \mu \frac{\partial u}{\partial y},$$

with a viscosity coefficient of order

$$\mu \sim \tfrac{1}{3}\,\rho\, v_{\mathrm{th}}\,\ell.$$

While this argument is heuristic, it captures the essential scaling and physical mechanism: viscosity is the diffusive transport of momentum due to microscopic motion. A more rigorous calculation using kinetic theory (e.g. the Chapman–Enskog solution for a dilute gas of hard spheres) yields a viscosity of the same order of magnitude, providing confirmation of this simple picture.

2.4.2 The Formal Theory of Viscosity

We now develop the formal framework for viscosity. Our ultimate goal is to build a constitutive law for viscous stresses that, when inserted into the momentum conservation equation, yields the celebrated Navier–Stokes equations.

The Viscous Stress Tensor

The central physical idea is that **viscosity encodes how *velocity gradients* generate internal stresses in a fluid**. Because viscosity represents a dissipative process, the stress must depend only on the *local deformation rate* of a fluid element.

The deformation of a fluid element is described by the velocity gradient tensor,

$$V_{\mu\nu} \equiv \frac{\partial u_\mu}{\partial x_\nu}.$$

This tensor can be decomposed into its symmetric and antisymmetric parts:

$$V_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu},$$

where $S_{\mu\nu}$ is the symmetric *rate-of-strain tensor* and $A_{\mu\nu}$ is the antisymmetric *vorticity tensor*. The latter corresponds to a rigid-body rotation, which does not distort the shape of the fluid element and therefore cannot produce internal stresses. Only the symmetric part $S_{\mu\nu}$ contributes to viscous effects.

Motivated by these considerations, we postulate the most general linear constitutive relation between stress and strain rate:

$$\tau_{ij} = \mu_{ij\mu\nu}\, S_{\mu\nu},$$

where τ_{ij} is the viscous stress tensor and $\mu_{ij\mu\nu}$ is the rank-4 *viscosity tensor* that encodes the proportionality between strain rates and stresses. In full generality, μ_{ijkl} could contain $3^4 = 81$ components in three dimensions. Physical symmetries reduce this number dramatically. There are a number of such symmetries; however, the following are the most relevant in the theory of viscous fluids:

1. Stress symmetry.

Because $\tau_{ij} = \tau_{ji}$, only the part of μ_{ijkl} symmetric under $i \leftrightarrow j$ contributes. Similarly, since S_{kl} is symmetric, only the part symmetric under $k \leftrightarrow \ell$ is relevant.

2. Isotropy.
An isotropic fluid must respond identically in all directions. Formally, this means μ_{ijkl} must remain invariant under any rotation:

$$\mu_{ijkl} = R_{ip}R_{jq}R_{kr}R_{\ell s}\,\mu_{pqrs}, \qquad \forall R \in SO(3).$$

The only building blocks of such isotropic tensors are Kronecker deltas. To construct a rank-4 isotropic tensor, we pair indices in all inequivalent ways.

3. Independent delta pairings.
There are exactly three independent pairings:

$$\delta_{ij}\delta_{k\ell}, \qquad \delta_{ik}\delta_{j\ell}, \qquad \delta_{i\ell}\delta_{jk}.$$

Thus the most general isotropic form is

$$\mu_{ijkl} = \alpha\,\delta_{ij}\delta_{k\ell} + \beta\,\delta_{ik}\delta_{j\ell} + \gamma\,\delta_{i\ell}\delta_{jk},$$

with scalar coefficients α, β, γ. (*Think about acting on the δ-based μ with the various rotation matrices. It is easy to convince yourself that this is one way to preserve the tensor. Then each of the possible ways to create a pairing can be present.*)

4. Reduction.
Contracting this form with S_{kl} yields

$$\tau_{ij} = \alpha\,\delta_{ij}S_{kk} + (\beta + \gamma)S_{ij}.$$

This shows that the viscous stress is determined by only two independent constants:

$$\tau_{ij} = 2\mu S_{ij} + \lambda\,\delta_{ij}S_{kk},$$

where $\mu = \beta + \gamma$ and $\lambda = \alpha$.

2.4.3 Shear and Bulk Viscosity

We have shown above that the viscous stress tensor takes the form

$$\tau_{ij} = 2\mu S_{ij} + \lambda\delta_{ij}S_{kk},$$

however, we may get an even more intuitive expression where things begin to reflect some degree of **physical structure**. We first recognize that S_{ij} has **diagonal elements** S_{ii} which correspond to **contraction and expansion**. Likewise, it has **off-diagonal elements** S_{ij} corresponding to **shearing**. What we can do is break $\boldsymbol{S}$ into these two parts:

1. The **isotropic component**: contains the uniform expansion / contraction of the material,
2. The **traceless (deviatoric) component**: which encodes the shear.

Formally, we write

$$S_{ij} = \underbrace{\frac{1}{3}S_{kk}\delta_{ij}}_{\text{Isotropic Component}} + \underbrace{\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)}_{\text{traceless component}}.$$

If we now propogate this expansion into the **viscous stress**, we find

$$\tau_{ij} = \underbrace{2\mu S'_{ij}}_{\text{Deviatoric Component}} + \underbrace{\left(\frac{2}{3}\mu + \lambda\right)\delta_{ij}S_{kk}}_{\text{Trace Component}}. \tag{2.9}$$

If we now combine our constants into a nice form, we can write

$$\boxed{\tau_{ij} = \underbrace{2\mu S'_{ij}}_{\text{Shear Viscous Stresses}} + \underbrace{\zeta\delta_{ij}S_{kk}}_{\text{Bulk Viscosity}},} \tag{2.10}$$

where we have introduced (or reinterpretted in the case of μ) the constants to be the coefficients of **bulk** and **shear** viscosity respectively. We now state this result with definitions:

Definition 2.4.1 — Shear Viscosity. The **shear viscosity** μ quantifies the internal resistance of a fluid to *shearing deformations*, i.e. motions where parallel fluid layers slide past one another at different velocities. It governs the stresses proportional to the traceless, symmetric part of the strain-rate tensor. Everyday examples include the drag experienced when stirring honey or the laminar shear between plates in Couette flow.

Definition 2.4.2 — Bulk Viscosity. The **bulk viscosity** λ quantifies the fluid's resistance to *volumetric deformations*, i.e. uniform compression or expansion. It appears as an isotropic stress proportional to $\nabla \cdot \mathbf{u}$. For incompressible flows ($\nabla \cdot \mathbf{u} = 0$), bulk viscosity plays no role. In compressible flows, howeverparticularly when internal degrees of freedom such as molecular vibrations or rotations are excitedbulk viscosity can strongly affect energy dissipation.

2.4.4 The Navier-Stokes Equations

We are now prepared to derive the most famous equation in fluid dynamics: the **Navier-Stokes equation**. This equation is the equivalent to the Euler equation when we include viscous stresses. Let's consider a system with **bulk viscosity** ζ and **shear viscosity** μ. The stress tensor $\boldsymbol{\sigma}$ will take the form

$$\boldsymbol{\sigma} = -P\mathbf{I} + \boldsymbol{\tau}.$$

Now, as we derived above (2.10),

$$\tau_{ij} = \underbrace{2\mu S'_{ij}}_{\text{Shear Viscous Stresses}} + \underbrace{\zeta\delta_{ij}S_{kk}}_{\text{Bulk Viscosity}},$$

This form is **not particularly conducive to manipulation**, as such, we'll use the form without splitting into the deviatoric and bulk terms, but this time, we'll replace λ with ζ:

$$\tau_{ij} = 2\mu S_{ij} + \left(\zeta - \frac{2}{3}\mu\right)\delta_{ij}S_{kk} = \mu(u_{i;j} + u_{j;i}) + \left(\zeta - \frac{2}{3}\mu\right)\delta_{ij}u_{k;k},$$

where the ; is the **covariant derivative**. In vector notation,

$$\boldsymbol{\tau} = \mu\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right] + \mathbf{I}\left(\zeta - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u}.$$

As such, we have

$$\sigma_{ij;j} = -P_{;i} + \tau_{ij;j} = -P_{;i} + \mu\left(u_{i;jj} + u_{j;ij}\right) + \left(\zeta - \frac{2}{3}\mu\right)\delta_{ij}u_{k;ki}.$$

Because the order of operations doesn't matter, this becomes

$$\sigma_{ij;j} = -P_{;i} + \tau_{ij;j} = -P_{;i} + \mu u_{i;jj} + \left(\zeta + \frac{1}{3}\mu\right)\delta_{ij}u_{k;ki}.$$

Note the manipulation of the last term... In vector notation, this is

$$\nabla\cdot\boldsymbol{\sigma} = -\nabla P + \mu\nabla^2\mathbf{u} + \left(1 + \frac{1}{3}\right)\nabla(\nabla\cdot\mathbf{u}).$$

Starting from the conservative momentum equation in terms of momentum density $\mathbf{p} = \rho\mathbf{u}$,

$$\frac{\partial\mathbf{p}}{\partial t} + \nabla\cdot(\mathbf{p}\otimes\mathbf{u}) + \nabla\cdot\boldsymbol{\sigma} = \mathbf{F}_{\text{ext}}, \tag{2.11}$$

Hence, we arrive at **the conservative form of the Navier–Stokes momentum equation** becomes

$$\boxed{\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla P + \mu\nabla^2\mathbf{u} + \left(\zeta + \tfrac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F}_{\text{ext}}.} \tag{2.12}$$

In its **convective (non–conservative) form**, we divide through by ρ and use the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla,$$

so that the momentum equation becomes

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{u} + \left(\zeta + \tfrac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F}_{\text{ext}}.} \tag{2.13}$$

In the limit of **incompressible flow**, the density is constant and the velocity field is solenoidal, $\nabla\cdot\mathbf{u} = 0$. In this case, the bulk viscosity term vanishes and the equation reduces to

$$\boxed{\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \mu\nabla^2\mathbf{u} + \mathbf{F}_{\text{ext}},} \tag{2.14}$$

subject to the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \tag{2.15}$$

Equations (2.12)–(2.14), together with the continuity equation and an appropriate energy equation or equation of state, constitute the **Navier–Stokes system** of fluid dynamics.

2.5 Surfaces and Boundaries

This needs to be written

Rigid boundaries (no-slip, slip).
Free surfaces (stress-free + material).
Fluidfluid interfaces (stress balance between two fluids).
Porous/permeable boundaries (flux across).
Phase-change interfaces (flux + stress balance).
Artificial/far-field boundaries (boundary conditions at infinity or domain edges).

3. Gravitation

R This chapter is largely subsumed by more invested literature; particularly the Binney & Tremmaine, which covers the details of gravitational fields in considerably greater detail. For that reason, most of the exposition of this chapter is skipped and should be reviewed via reference to that text.

3.1 The Virial Theorem

One element of classical mechanics of extreme importance for fluid dynamics is the **virial theorem**, which describes the distribution of energy between potential and kinetic energy in a classical system. We will use it for many computations in astrophysical fluid dynamics.

3.1.1 Dilation Asymmetry

The core result of the virial theorem does not really depend on its origin; however, there is some interesting classical physics driving this result which can be interesting. We include this section here to cover this interesting physics; however, it may be skipped without loss of continuity.

Consider a generic classical system subject to a Lagrangian L of the form

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i) = \frac{1}{2}\sum_i m_i \dot{r}_i^2 - V(\mathbf{r}).$$

We might consider the behavior of the system under an **infinitesimal dilation**. Formally, we consider a transformation

$$\mathbf{r}_i \to (1+\epsilon)\mathbf{r}_i.$$

The resulting change in the action integrand will be

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \mathbf{r}_i} \cdot \delta \mathbf{r}_i + \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \delta \dot{\mathbf{r}}_i \right) = \sum_i \left(-\frac{\partial V}{\partial \mathbf{r}_i} \cdot \mathbf{r}_i + m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \epsilon$$

$$\delta L = \left(2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right) \epsilon$$

Now, symmetry only occurs when

$$2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i = 0,$$

which is not a standard occasion. Thus, the infinitesimal dilation is **not** a symmetry of the Lagrangian unless the above condition is satisfied at every instant in time. For most potentials, this is not generally the case. However, this lack of symmetry is not useless instead, it gives rise to a physically meaningful quantity whose dynamics encode the failure of dilation symmetry.

To further understand this, consider the quantity

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i,$$

which we interpret as the **Noether-like charge** associated with dilation. It is not generally conserved, but it plays an important role in encoding the system's behavior under spatial rescaling.

Taking its time derivative:

$$\frac{dG}{dt} = \sum_i (\dot{\mathbf{p}}_i \cdot \mathbf{r}_i + \mathbf{p}_i \cdot \dot{\mathbf{r}}_i) = \sum_i (\mathbf{F}_i \cdot \mathbf{r}_i + m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i + 2T$$

This equation explicitly shows how the rate of change of the dilation generator G is governed by the degree to which the forces are not scale-invariant. Thus, in a scale invariant system, $dG/dt = 0$. Although G is not conserved instantaneously, we may consider its time average for systems in a steady or bounded configuration. Assuming the system does not expand or contract indefinitely, the long-term average of $\frac{dG}{dt}$ vanishes:

$$\left\langle \frac{dG}{dt} \right\rangle = 0 \quad \Rightarrow \quad \left\langle 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle = 0.$$

This is the **virial theorem**, recovered as the time-averaged failure of dilation symmetry. It shows that, for a system in equilibrium or exhibiting periodic motion, the kinetic energy is balanced by the potential's response to scaling:

$$\boxed{\langle T \rangle = -\frac{1}{2} \left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle}$$

3.1.2 For Gravitational Systems

Consider that the above system is actually a set of particles with positions $\mathbf{r}_i$ which are interacting gravitationally. In such a case, we have

$$\mathbf{F}_i = -G \sum_{i \neq j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} (\mathbf{r}_i - \mathbf{r}_j).$$

Thus,

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -G \sum_i \sum_{i \neq j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} (\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{r}_i.$$

For a given pair i, j, we have terms

$$-G \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} |\mathbf{r}_i - \mathbf{r}_j|^2 = -G \frac{m_i m_j}{r_{ij}}.$$

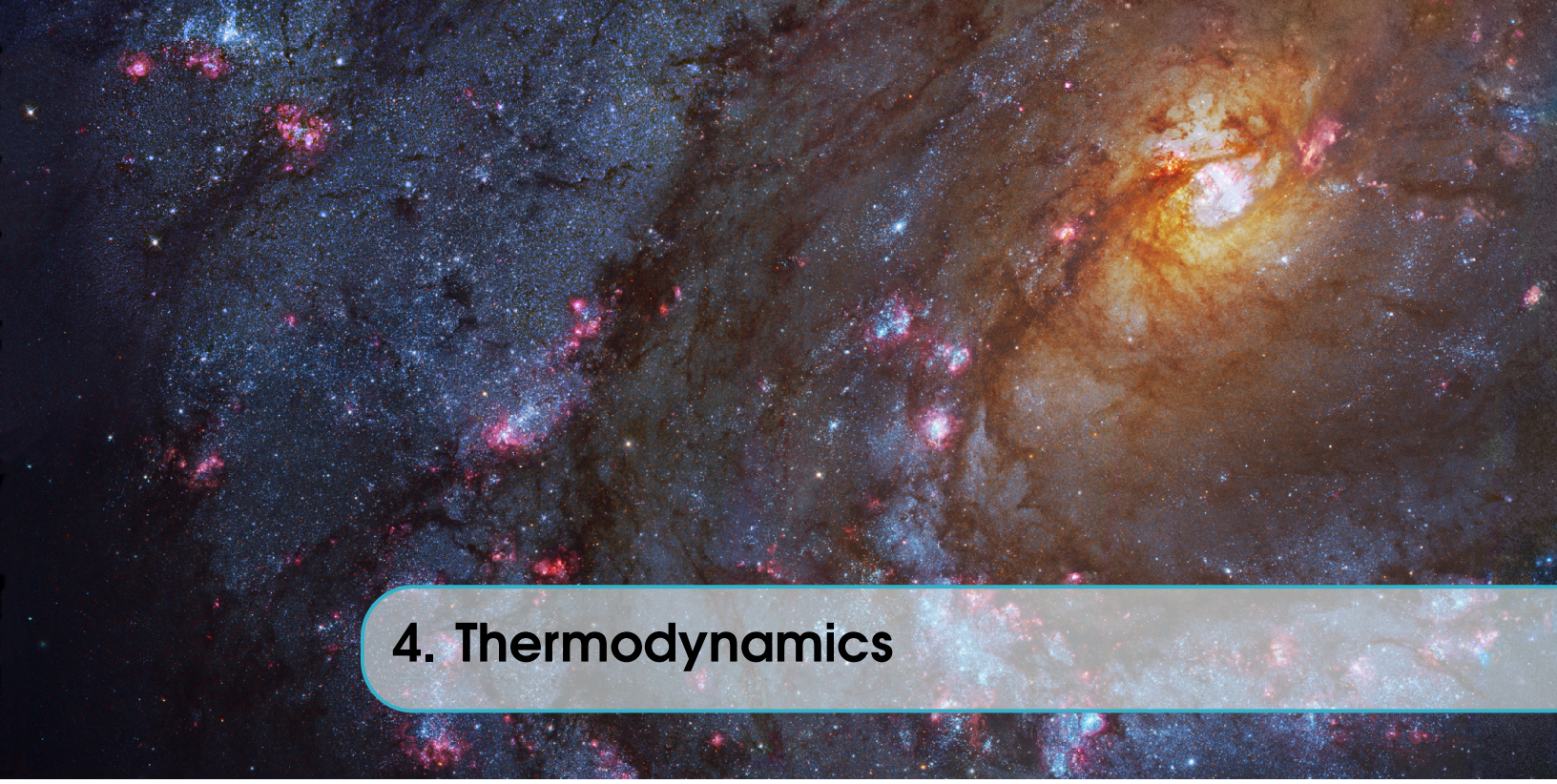
Thus,

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = V,$$

where V is the **gravitational potential**. Thus, we have the relationship that

$$\boxed{\langle V \rangle + 2 \langle T \rangle = 0.}$$

where $\langle T \rangle$ is the **total kinetic energy** and $\langle V \rangle$ is total **potential energy**.



4. Thermodynamics

Thus far, we have derived two equations (the **Euler Equations**) which involve several fluid variables:

1. **Conservation of Mass (Continuity Equation)**:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$
2. **Conservation of Momentum**:
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F}_{\text{ext}}.$$

Assuming the external force field $\mathbf{F}_{\text{ext}}$ is known from the problem setup (e.g., gravity), the system contains **three primary unknowns**: the mass density ρ, the velocity field $\mathbf{u}$, and the pressure p. With only two equations, this system is **underdetermined**, and we require an additional relation to close it.

Idea

If the fluid admits a thermodynamic description, we may posit an **equation of state** (EOS) of the form

$$F(\rho, p, u) = 0,$$

where u is the internal energy per unit mass. This provides a link between thermodynamic variables but does not by itself close the system. To proceed, we generally take one of the following approaches:

1. **Include an energy equation**: Add a conservation law for internal energy or entropy to evolve u dynamically. This is required in fully general, compressible, thermodynamic flows.
2. **Assume a limiting thermodynamic regime**: Impose a simplification based on physics (e.g., rapid cooling or adiabatic expansion) that reduces the EOS to a barotropic form:
$$p = p(\rho),$$
thereby eliminating u as an independent variable.

R Above, we assert that the fluid permits a **thermodynamic description**, which is a common scenario, but not one to be taken for granted. In order for a fluid to be described by equilibrium thermodynamics, we require **Local Thermodynamic Equilibrium**. Formally, a fluid element with characteristic length scale δL must be able to reach a maximum entropy state much more quickly than any dynamical changes can occur.

The ability for a maximum entropy state to be reached is dependent on the interaction scale (**the mean free path**) between collisions λ_{mfp}. If $\lambda_{\text{mfp}} \ll \delta L$, then **interactions are localized**, which is a critical element of our thermodynamic assumption. Likewise, if t_{coll} is the mean **collision time**, then $t_{\text{coll}} \ll t_{\text{dyn}}$ implies that the system has time to equilibrate much faster than its dynamical timescale.

In the following sections, we will explore our options for treating these thermodynamically equilibrated systems.

4.1 Important Elements of Thermodynamics

Before we continue with our treatment of equations of state, we will first review a few important elements of thermodynamics. Specifically, there are a few words to be had regarding the notion of entropy, and of adiabatic processes which is relevant to our continued discussion.

4.1.1 Equipartition

The **equipartition theorem** states that each quadratic degree of freedom in a system contributes $\frac{1}{2}kT$ to the mean energy. For a system of N particles with f quadratic degrees of freedom each, the total internal energy is therefore

$$U = \frac{f}{2}NkT.$$

4.1.2 Specific Heats and the Adiabatic Index

The notion of specific heat arises naturally once we combine the first law of thermodynamics with equipartition. The specific heat is defined as the amount of heat required to change the temperature of a system,

$$C \equiv \frac{dQ}{dT},$$

and its value depends on the thermodynamic constraint under which the process occurs. From the first law,

$$dU = dQ - PdV,$$

and using the equipartition expression for internal energy,

$$dU = \frac{f}{2}Nk\,dT,$$

we can directly relate changes in heat to changes in temperature.

If the volume is held fixed, the work term vanishes and one finds

$$C_V = \left(\frac{dQ}{dT}\right)_V = \frac{f}{2}Nk,$$

which reflects the fact that all the supplied heat goes into raising the internal energy. On the other hand, if the pressure is held constant, then the system must also do expansion work as the temperature changes. In this case,

$$\left(\frac{dQ}{dT}\right)_P = \frac{f}{2}Nk + P\frac{dV}{dT}.$$

Invoking the ideal gas law $PV = NkT$, we find that $P(dV/dT) = Nk$, and thus

$$C_P = \left(\frac{f}{2} + 1\right)Nk.$$

The ratio of these two heat capacities defines the **adiabatic index**,

$$\gamma \equiv \frac{C_P}{C_V} = 1 + \frac{2}{f}.$$

This simple relation shows how γ is determined entirely by the number of quadratic degrees of freedom of the particles. For a monatomic ideal gas with $f = 3$, we obtain $\gamma = 5/3$, while for a diatomic gas at moderate temperatures with $f = 5$, we have $\gamma = 7/5$. The adiabatic index is of central importance in fluid dynamics, since it governs the speed of sound in an ideal gas and dictates the relation between pressure and density during adiabatic processes.

4.1.3 Entropy in a Fluid

In a formal sense, the **SackurTetrode equation** gives the entropy of an ideal gas; however, this form is often unwieldy in fluid dynamics, since it encodes the full quantumstatistical information. Instead, we can work more gently from the thermodynamic identities.

Starting from the first law per unit mass,

$$du = T\,ds - P\,d\left(\frac{1}{\rho}\right) = T\,ds - \frac{P}{\rho^2}d\rho,$$

we obtain

$$ds = \frac{1}{T}\left(du + \frac{P}{\rho^2}d\rho\right).$$

For an ideal gas with f quadratic degrees of freedom,

$$u = c_v T, \qquad du = c_v\,dT,$$

where the specific heat at constant volume is

$$c_v = \frac{f}{2}\frac{k}{\mu m_p}.$$

Thus,

$$ds = \frac{1}{T}\left(c_v\,dT + \frac{P}{\rho^2}d\rho\right).$$

Using the ideal gas law,

$$P = \frac{\rho k T}{\mu m_p} = \rho R T,$$

with $R = k/(\mu m_p)$, we have

$$\frac{dP}{P} = \frac{dT}{T} + \frac{d\rho}{\rho}.$$

Substituting for dT/T gives

$$ds = c_v\left(\frac{dP}{P} - \frac{d\rho}{\rho}\right) + R\frac{d\rho}{\rho}.$$

Collecting terms,

$$ds = c_v \frac{dP}{P} - (c_v + R)\frac{d\rho}{\rho} = c_v \frac{dP}{P} - c_p \frac{d\rho}{\rho},$$

where $c_p = c_v + R$ and $\gamma = c_p/c_v$. Integration then yields

$$\boxed{s = c_v \ln\left(\frac{P}{\rho^\gamma}\right) + \text{const.}} \tag{4.1}$$

Big Idea

We should therefore **recognize** that the combination

$$\frac{P}{\rho^\gamma} \quad \propto \quad e^{s/c_v}$$

serves as a practical **entropy proxy**. In fact, when an equation of state is written in polytropic form,

$$P = K\rho^\gamma,$$

the constant K is directly related to the entropy, with $K \propto e^{s/c_v}$.

4.1.4 Isentropic Processes

An important special case in fluid dynamics is the **isentropic process**, in which the entropy remains constant. Since for an ideal gas we have

$$s = c_v \ln\left(\frac{P}{\rho^\gamma}\right) + \text{const},$$

holding s fixed implies that the combination P/ρ^γ must also remain constant. Thus, in an isentropic transformation, the pressure and density are related by the **polytropic equation of state**

$$P = K\rho^\gamma,$$

where the constant K encodes the entropy of the system. In other words, isentropic processes are exactly those in which the polytropic constant does not change.

The relevance of this result is considerable. In the absence of shocks or dissipation, adiabatic flows in gases are also isentropic flows, and so the condition $P \propto \rho^\gamma$ holds along streamlines. This relation underpins many results in fluid dynamics: it sets the speed of sound in an ideal gas,

$$c_s = \sqrt{\frac{\gamma P}{\rho}},$$

it determines the pressuredensity relation inside stars and polytropic models of stellar structure, and it is the key assumption in many treatments of compressible flow, including shock tubes, nozzle flows, and astrophysical accretion. In short, the isentropic assumption allows the fluid equations to be closed with a simple but powerful equation of state, tying thermodynamic structure directly to dynamical behavior.

4.1.5 Thermodynamic Potentials

A key theme in thermodynamics is the pairing of **extensive variables**, which scale with system size, and their conjugate **intensive variables**, which are independent of system size. Energy is always expressed in terms of these conjugate pairs, for example

$$dU = T\,dS - P\,dV + \mu\,dN,$$

where (S, V, N) are extensives and $(T, -P, \mu)$ are their conjugates. The extensives can be thought of as the natural "coordinates" of the system, while the intensives measure how the energy changes when those coordinates are varied.

There are four **standard thermodynamic potentials**, each adapted to different natural variables. Their definitions and uses are summarized below:

Potential	Natural Variables	Definition
Internal Energy U	(S, V, N)	$U(S, V, N)$
Helmholtz Free Energy F	(T, V, N)	$F = U - TS$
Enthalpy H	(S, P, N)	$H = U + PV$
Gibbs Free Energy G	(T, P, N)	$G = U - TS + PV$

Each potential is related to the others by a Legendre transform, which swaps an extensive variable for its conjugate intensive one. In this sense, choosing a potential is much like choosing a different coordinate chart for the thermodynamic state space.

Among these, **enthalpy** plays a particularly important role in fluid dynamics. Because many flows and reactions occur at constant pressure rather than constant volume, it is convenient to use $H(S, P, N)$, whose differential is

$$dH = T\,dS + V\,dP + \mu\,dN.$$

This form makes pressure, rather than volume, the natural variable, which aligns well with practical conditions such as atmospheric pressure in laboratory or astrophysical systems. As we will see later, enthalpy appears naturally in conservation laws and in the description of compressible flows. Now, by the first law of thermodynamics,

$$du = Tds - Pd\frac{1}{\rho} \implies \boxed{dh = Tds + \frac{1}{\rho}dP}.$$

Why is this helpful? It is helpful because, in **barotropic equations of state**, where $h(p)$ is a function of the pressure (or density) only, we have

$$dh = \frac{1}{\rho}dP.$$

Formally, the Tds doesn't go away, but we are constrained to take a path respecting the EOS, which means that we have entropy path independence. As such, instead of

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla P}{\rho}, \text{we have, } \frac{D\mathbf{u}}{Dt} = -\nabla h.$$

Thus, h **has a particular use as a potential of the fluid in the Euler equations**.

4.2 Equations of State

Formally, we **define** an equation of state such that

Definition 4.2.1 — Equation of State. Given some set of **complete thermodynamic variables** (i.e. variables which span the phase space) $(X_1, X_2, \ldots)$, an **equation of state** is a relationship between these thermodynamic variables of the form

$$F(X_1, \ldots, X_n) = 0. \tag{4.2}$$

In general, we commonly see equations of state for (p, ρ, T) or (p, ρ, u).

There are a number of common equations of state which are relevant for representing different physics. The most well known of these is the **ideal gas equation of state**:

Definition 4.2.2 — Ideal Gas. An **ideal gas** is a theoretical gas composed of a large number of identical, point-like particles that interact only via elastic collisions and do not exert long-range forces on one another. The behavior of an ideal gas is governed by the **ideal gas law**:

$$pV = Nk_BT,$$

where:

- p is the pressure of the gas,
- V is the volume it occupies,
- N is the total number of particles,
- k_B is the Boltzmann constant,
- T is the absolute temperature of the gas.

In astrophysical cases, we usually end up using

$$n = \frac{N}{V} = \frac{\rho}{m_p\mu},$$

where μ is the **mean particulate weight** of a given particle species in the gas. In this form, we have

$$\boxed{p = \frac{\rho kT}{m_p\mu} \sim \omega\rho T.} \tag{4.3}$$

R While most astrophysical fluids are well approximated by the ideal gas formalism, there are some fluids which require a more robust treatment. These result from a number of characteristic limits on the ideal gas that are worth being familiar with:

1. **High Temperatures:** In very high temperature scenarios, the kinetic energy of the particles leads to speeds at a significant fraction the speed of light and therefore requires relativistic treatment. This formally occurs when $kT \simeq mc^2$ and the energy stored in the mass is similar to the kinetic energy. In these cases, relativistic equations of state are required (i.e. $p = \rho c^2/3$ for ultra-relativistic gasses).

2. **Strong Coupling:** In some cases, electromagnetic interactions occur over longer ranges and induce non-ideal behavior. This then becomes the realm of plasma physics (in most cases). We generally see this in stellar interiors, but it is becoming more relevant in many of the diffuse astrophysical plasmas we consider.

4.2.1 Isothermal Equations of State

Let's return to the issue of the temperature. In general, we need a prescription for how the temperature behaves in the fluid using some sort of equation for **energy transport**, which can often be quite difficult. In some cases, we might invoke an approximate rule for the temperature which is founded in a thermodynamic assumption. The resulting EOS is called a **barotropic equation of state** and there are generally two cases that get considered.

In an **isothermal** gas, the temperature is held **fixed** throughout the evolution of the system. As a result, the equation of state reduces to a linear relation between pressure and density:

$$p = \frac{\rho k_B T}{\mu m_p} = K\rho,$$

where:

- k_B is Boltzmann's constant,
- μ is the mean molecular weight,
- m_p is the proton mass,
- T is the (constant) temperature,
- $K = \frac{k_B T}{\mu m_p}$ is a constant of proportionality.

This linear relation between pressure and density effectively **closes** the fluid equations without requiring an additional energy or entropy equation.

The **isothermal approximation** is valid when each fluid element maintains approximately the same temperature over its evolution. Physically, this requires that any thermal energy gained (due to compression, shocks, or other dynamical processes) is quickly radiated or otherwise dissipated before the temperature can change appreciably.

R In a typical dynamical time interval δt, a fluid element may undergo motion, compression, shocks, or other interactions that introduce thermal energy these are the **dynamical processes**. Simultaneously, the fluid may also lose energy through radiation, turbulent dissipation, conduction, or viscosity these are the **cooling processes**. The isothermal approximation is justified when the cooling timescale $t_{\rm cool}$ is much shorter than the dynamical timescale $t_{\rm dyn}$, i.e.,

$$t_{\rm cool} \ll t_{\rm dyn}.$$

In this regime, any heating is rapidly counteracted by efficient cooling, and the temperature remains effectively constant in time along the path of a fluid element.

Note that this condition applies *locally* along the trajectories of individual fluid parcels; it does not require the temperature to be globally uniform throughout the fluid domain. Therefore, isothermal models may still allow for slow or large-scale spatial variation in temperature, provided the temperature is approximately constant *along* each fluid element's motion.

The following are classic cases where these approximations can be applied:

1. **Molecular Cloud Cores (Pre-Stellar Collapse)**
   In the early stages of star formation, molecular clouds are cold ($T \sim 10$ K) and radiate efficiently via dust and molecular line emission (e.g., CO, CII). The cooling time is much shorter than the free-fall time, making the temperature nearly constant during collapse. Hence, isothermal models are often used to study fragmentation, gravitational instability (e.g., Jeans analysis), and early core dynamics.
2. **Thin Accretion Disks (Vertical Structure)**
   In geometrically thin disks (e.g., around stars or black holes), the vertical thermal timescale is much shorter than the radial viscous timescale. This allows for rapid radiative relaxation in the vertical direction, justifying a locally isothermal assumption ($T = T(r)$, constant in z) when analyzing hydrostatic balance and vertical disk structure.
3. **Self-Gravitating Isothermal Spheres (e.g., BonnorEbert Spheres)**
   These models describe equilibrium structures of self-gravitating gas in thermal contact with a surrounding reservoir. The assumption of an externally regulated or thermostatted temperature yields simple equilibrium equations and captures the onset of gravitational instability in pressure-confined clouds.
4. **Idealized Jeans Instability Calculations**
   In linear perturbation theory, the isothermal EOS simplifies the analysis of gravitational collapse. It removes complications from entropy or energy evolution, focusing purely on the balance between self-gravity and thermal pressure.
5. **Numerical Simulations with Rapid Cooling**
   In some hydrodynamical simulations (e.g., cosmological structure formation, protoplanetary disks), the gas is assumed to cool instantaneously or follow a prescribed temperature profile. This justifies an isothermal approximation when detailed radiative transfer or energy evolution is computationally prohibitive.
6. **Planetary Atmospheres (Isothermal Layers)**
   Upper layers of planetary atmospheres, where radiative equilibrium is maintained with a surrounding radiation field, can often be modeled as isothermal. This is especially useful in analytic modeling of scale heights and hydrostatic profiles.

4.2.2 Adiabatic Equations of State

Consider an ideal gas which undergoes processes quickly enough that no heat is able to enter or leave the fluid element. In such a scenario, we have the **ideal gas equation of state**

$$p = \frac{\rho k T}{m_p \mu},$$

and we have also the **first law of thermodynamics** (in terms of quantities per unit mass), which takes the form

$$\delta \tilde{U} = -p d\tilde{V} = \frac{p}{\rho^2} d\rho$$

in the absence of heat transfer. *We use $\tilde{}$ to denote quantities per unit mass.* By **equipartition**, we also have

$$\tilde{U} = \frac{f}{2} \frac{kT}{m_p \mu} \implies \delta \tilde{U} = \frac{f}{2} \frac{k dT}{m_p \mu}.$$

If we equate these two statements, we have

$$\frac{p}{\rho^2}d\rho = \frac{f}{2}\frac{kdT}{m_p\mu}.$$

Utilizing the equation of state,

$$\frac{m_p\mu}{k}\left[\frac{dp}{\rho} - \frac{p}{\rho^2}d\rho\right] = dT \implies \frac{p}{\rho^2}d\rho = \frac{f}{2}\left[\frac{dp}{\rho} - \frac{p}{\rho^2}d\rho\right].$$

We therefore may write the differential equation as

$$\frac{f+2}{f}\frac{d\rho}{\rho} = \frac{dp}{p} \implies p \propto \rho^\gamma,$$

where $\gamma = (f+2)/f$ is the **adiabatic index**. This **polytropic relation** between pressure and density closes the system of fluid equations without requiring a separate temperature or entropy evolution law, provided the process remains adiabatic.

The **adiabatic approximation** is valid when fluid elements evolve on timescales short enough that there is negligible radiative energy exchange or dissipation. In particular, the entropy of each parcel remains constant:

$$\frac{ds}{dt} = 0.$$

This assumption holds in dynamically evolving systems where thermal transport processes are slow compared to bulk motion such as in rapidly collapsing, expanding, or oscillating fluids.

R In a dynamical time interval δt, a fluid parcel may be compressed or expanded, doing work and changing its internal energy. If this happens faster than energy can be transported via radiation, conduction, or viscosity, then the evolution is **adiabatic**. Additionally, if shocks or turbulence are weak or absent, the process remains **isentropic**, meaning that the fluid retains constant entropy. This regime is described by the adiabatic (isentropic) equation of state:

$$p \propto \rho^\gamma.$$

Note that K can vary from one fluid element to another if entropy varies, but in isentropic flow, K is constant along fluid paths. Therefore, this EOS captures compressional heating and expansion cooling, unlike the isothermal case.

The following are classic contexts where the isentropic approximation is applied:

1. **Stellar Interiors (Convective Regions)**
   In regions of efficient convection, heat is transported by fluid motion much faster than by radiation. The rising and sinking fluid parcels experience nearly adiabatic processes, justifying an isentropic EOS in local models of stellar interiors.
2. **Adiabatic Collapse in Star Formation**
   During later stages of protostellar collapse, the central regions of molecular clouds become optically thick, preventing efficient cooling. Compressional heating dominates, and the gas heats adiabatically. This is critical in setting the first hydrostatic core conditions.

3. **Idealized Hydrodynamics and Sound Waves**
   Many textbook problems (e.g., wave propagation, Riemann problems, acoustic oscillations) assume isentropic flow to simplify the analysis and isolate compressibility effects without heat exchange.
4. **Neutrino-decoupled Core Collapse Supernovae (Early Phase)**
   In the early, adiabatically collapsing phase of a supernova core (prior to shock breakout or neutrino trapping), entropy remains approximately constant within shells. Isentropic models capture the stiffening of the EOS due to compression.
5. **Polytropic Models of Stars and Planets**
   Many equilibrium models for stars and planets use polytropic equations of state with fixed γ to approximate pressure support against gravity, especially when entropy gradients are weak.
6. **Ballistic or Rarefied Flows (e.g., Outflows, Ejecta)**
   In expanding flows where radiative losses are negligible (e.g., outer supernova ejecta, stellar winds in the adiabatic zone), entropy is conserved and the flow is well-described by an isentropic EOS.

4.2.3 Polytropic Equations of State

A wide class of simple closures for the equation of state take the form

$$p = K\rho^{\Gamma},$$

for some constant K and exponent Γ. Special cases of this relation include

- $\Gamma = 1$: the **isothermal** form, $p \propto \rho$, which matches the pressure–density relation of an ideal gas at constant temperature,
- $\Gamma = \gamma$: the **adiabatic** or **isentropic** form, corresponding to an ideal gas with constant entropy.

More generally, we may use this as a **phenomenological equation of state** even when the underlying microphysics is unknown. Suppose the true EOS has some complicated form $F(p, \rho, u) = 0$; we then approximate it by a polytropic relation of the above type. In this case, the thermodynamic state is entirely specified by ρ, so the specific internal energy u is a function of density alone. From the first law,

$$du = \frac{p}{\rho^2}\, d\rho = \frac{K}{\rho^2}\rho^{\Gamma} d\rho = K\rho^{\Gamma-2}\, d\rho,$$

which integrates to

$$u(\rho) = \frac{K}{\Gamma - 1}\, \rho^{\Gamma-1} + \text{const}, \qquad (\Gamma \neq 1).$$

Similarly, the specific enthalpy $h = u + \frac{p}{\rho}$ is

$$h(\rho) = \frac{\Gamma}{\Gamma - 1} K\rho^{\Gamma-1}.$$

R It is important to emphasize that adopting $p = K\rho^{\Gamma}$ as a closure relation does not, by itself, imply anything about the microphysics.

- If $\Gamma = 1$, this does not mean the fluid is *physically isothermal*; it only means that the p–ρ relation coincides with that of an ideal gas at constant temperature.
- If $\Gamma = \gamma$, it does not prove the system is *truly isentropic*; it only reproduces the form expected of an adiabatic ideal gas.

In all cases, the polytropic EOS is best understood as a barotropic closure: $p = p(\rho)$. This assumption eliminates entropy as an independent thermodynamic variable, and so necessarily sacrifices information about the detailed microphysics of the fluid.

4.2.4 Summary

Before moving on, it is worth taking stock of what has been accomplished by the use of **barotropic equations of state**. We have identified two cases which are determined on the basis of the relevant time scales:

1. ($t_{\text{cool}} \gg t_{\text{dyn}}$): We have efficient cooling which outweighs the influence of any dynamical effects. Therefore, we can treate the fluid as **isothermal**.
2. ($t_{\text{dyn}} \ll t_{\text{cool}}$): We have dynamical changes (pressure changes) which influence the fluid on time scales which are far too short for effective heat transfer. Therefore, we can use the **adiabatic approximation**.

As such, we see that our two barotropic approximations are really two sides of a scale balancing cooling and dynamical effects. We can therefore address any problem in astrophysical fluid dynamics by first examining what processes are dominant: cooling or dynamics.

4.3 The Energy Equation

So far in this chapter of the notes, we've discussed only the details of **phenomenological equations of state**, where we can reduce the thermodynamic phase space to a single variable (generally ρ). What if we have a general equation of state, which (critically), does **not provide closure**? In this scenario, we need to be considerably more concrete about our treatment and instead introduce an entire equation describing the energy transport. In this section, we'll treat the details of this approach and introduce the last of our critical equations: the **Energy Equation**.

4.3.1 The Internal Energy

Consider a single fluid element in a fluid. It is subject to the first law of thermodynamics, which requires that the energy per unit mass behave as

$$d\epsilon = dq + \frac{P}{\rho^2} d\rho,$$

where dq is the **heating per unit mass**, and the second term is the work done by the system. Because we are working in the **Lagrangian Frame**, we may describe the change of ϵ with time as

$$\boxed{\frac{D\epsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} - \dot{q}_{\text{cool}},} \tag{4.4}$$

where now $\dot{q}_{\text{cool}}$ is the rate at which energy **leaves the fluid** due to cooling. This is effectively the **first law of thermodynamics** as applied to a single fluid element.

4.3.2 Total Energy

Consider the **total energy density**

$$\mathcal{E} = \rho\left(\frac{1}{2}u^2 + \Phi + \epsilon\right).$$

The resulting time derivative is going to be

$$\frac{D\mathcal{E}}{Dt} = \frac{D}{Dt}\left(\rho\left[\frac{1}{2}u^2 + \Phi + \epsilon\right]\right).$$

Applying the product rule yields

$$\frac{D\mathcal{E}}{Dt} = \frac{\mathcal{E}}{\rho}\frac{D\rho}{Dt} + \rho\left(\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \frac{D\Phi}{Dt} + \frac{D\epsilon}{Dt}\right).$$

Now, we can exploit several instances of known quantities in order to manipulate this to our designs. Most usefully, we will use the **continuity equation** and the **Euler equation** to get rid of some extraneous variables:

$$\frac{D\mathcal{E}}{Dt} = \frac{\mathcal{E}}{\rho}\underbrace{\frac{D\rho}{Dt}}_{\text{continuity}} + \rho\left(\mathbf{u}\cdot\underbrace{\frac{D\mathbf{u}}{Dt}}_{\text{Euler}} + \frac{D\Phi}{Dt} + \frac{D\epsilon}{Dt}\right).$$

From this, we see that

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\,\nabla\cdot\mathbf{u} + \rho\left(\mathbf{u}\cdot\left[-\frac{\nabla P}{\rho} - \nabla\Phi\right] + \frac{D\Phi}{Dt} + \frac{D\epsilon}{Dt}\right)$$

Substituting for the Euler equation, the continuity equation, and expanding the gravitational derivative, we obtain

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\,\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla P - \rho\,\mathbf{u}\cdot\nabla\Phi + \rho\left(\frac{\partial\Phi}{\partial t} + \mathbf{u}\cdot\nabla\Phi\right) + \rho\frac{D\epsilon}{Dt} \tag{4.5}$$

$$= -\mathcal{E}\,\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla P + \rho\frac{\partial\Phi}{\partial t} + \rho\frac{D\epsilon}{Dt}. \tag{4.6}$$

Now, invoking the first law of thermodynamics in the Lagrangian frame [Eq. (4.4)],

$$\rho\frac{D\epsilon}{Dt} = \frac{P}{\rho}\frac{D\rho}{Dt} - \rho\dot{q}_{\rm cool} = -P\nabla\cdot\mathbf{u} - \rho\dot{q}_{\rm cool},$$

where we used the continuity equation in the final step.

Thus the total energy equation becomes

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\,\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla P - P\nabla\cdot\mathbf{u} + \rho\frac{\partial\Phi}{\partial t} - \rho\dot{q}_{\rm cool}. \tag{4.7}$$

Notice that the two pressure terms can be combined:

$$-\mathbf{u}\cdot\nabla P - P\nabla\cdot\mathbf{u} = -\nabla\cdot(P\mathbf{u}),$$

so the Lagrangian form simplifies to

$$\boxed{\frac{D\mathcal{E}}{Dt} + \mathcal{E}\,\nabla\cdot\mathbf{u} + \nabla\cdot(P\mathbf{u}) = \rho\frac{\partial\Phi}{\partial t} - \rho\dot{q}_{\rm cool}.} \tag{4.8}$$

4.3.3 Eulerian Conservation Form

By converting the Lagrangian energy equation into Eulerian form, we obtain

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot [(\mathcal{E} + P)\,\mathbf{u}] = \rho \frac{\partial \Phi}{\partial t} - \rho \dot{q}_{\text{cool}}.} \tag{4.9}$$

This is the Eulerian conservation-law form of the total energy equation, with flux

$$\mathbf{F}_{\mathcal{E}} = (\mathcal{E} + P)\,\mathbf{u} = \rho \left(\tfrac{1}{2}u^2 + \Phi + h\right)\mathbf{u},$$

where $h = \epsilon + P/\rho$ is the specific enthalpy.

In the absence of time-dependent gravitational fields and cooling, the conservation law reduces to

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left[\rho \left(\tfrac{1}{2}u^2 + \Phi + h\right)\mathbf{u}\right] = 0,$$

which shows that the quantity

$$B = \tfrac{1}{2}u^2 + \Phi + h$$

acts as an **energy potential**. This is exactly the **Bernoulli function**, familiar from Bernoullis theorem.

> R One can therefore view the Eulerian energy equation as nothing more than the local conservation of Bernoullis function: energy is transported in the flow via the flux of B. This provides a useful mnemonic: the entire equation can be remembered as a generalization of Bernoullis theorem to time-dependent, compressible flows with sources and sinks.

4.4 Energy Transport

We have now achieved the primary goal of this section: **deriving all 3 fundamental equations** of fluid dynamics.

Summary of Ideal Fluid Equations

1. Continuity Equation (Mass Conservation)

$$\text{Lagrangian:} \quad \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$
$$\text{Eulerian:} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

2. Momentum Equation (Euler's Equation)

$$\text{Lagrangian:} \quad \rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{f}_{\text{ext}}$$
$$\text{Eulerian:} \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f}_{\text{ext}}$$

3. Energy Equation (Total Energy Conservation)

$$\text{Lagrangian:} \quad \frac{D\mathcal{E}}{Dt} + \mathcal{E}\,\nabla \cdot \mathbf{u} + \nabla \cdot (P\mathbf{u}) = \rho \frac{\partial \Phi}{\partial t} - \rho \dot{q}_{\text{cool}}$$
$$\text{Eulerian:} \quad \frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] = \rho \frac{\partial \phi}{\partial t} - \rho \dot{q}_{\text{cool}}$$

We are still left with a serious quandry: **what is $\dot{q}_{\text{cool}}$?** That is the question that we will address in this section.

4.4.1 Cosmic Ray Heating

Cosmic rays are **ultra-relativistic** particles (primarily protons and heavier nuclei) produced in high-energy astrophysical processes such as supernova shocks or active galactic nuclei. They propagate throughout the interstellar medium (ISM), often with energies far exceeding 1 MeV, and can penetrate deep into shielded regions such as molecular clouds where UV photons cannot reach.

When a cosmic ray collides with a **neutral hydrogen atom**, it can ionize the atom, producing a free electron with substantial kinetic energy. This high-energy **secondary electron** subsequently interacts with the surrounding medium and deposits energy through several possible channels:

- **Secondary Ionization**: The energetic electron may ionize additional atoms or molecules (especially in mostly neutral environments).
- **Excitation**: It may excite atoms or molecules, which subsequently de-excite and emit photons (often lost from the system).
- **Elastic (Coulomb) Collisions**: It may scatter off ambient electrons and ions, transferring energy to thermal motionthis constitutes **heating**.
- **Bremsstrahlung Radiation**: Emission from electron deceleration, typically a subdominant cooling channel in cold gas.

In practice, rather than track all these processes in detail, we parameterize the effect by assuming that each **primary ionization** event deposits a mean thermal energy ΔE_{heat}

into the gas. The total **volumetric heating rate** is then given by:

$$\Gamma_{\rm CR} = n_{\rm H}\,\zeta_{\rm CR}\,\Delta E_{\rm heat}$$

where:

- $n_{\rm H}$: Number density of hydrogen atoms [$\rm cm^{-3}$],
- $\zeta_{\rm CR}$: Cosmic ray ionization rate per H atom [$\rm s^{-1}$],
- $\Delta E_{\rm heat}$: Mean thermal energy deposited per ionization [erg] (typically $\sim 1020\,\rm eV$).

This expression captures the essential thermal impact of cosmic rays in dense or UV-shielded astrophysical environments, where they often dominate over radiative heating sources.

Idea

Note that these scale with the density **linearly**, so they are generally sub-dominant processes in most scenarios.

4.4.2 Thermal Conduction

Thermal conduction is the process by which heat energy is transferred through a medium due to random microscopic motion of particles primarily electrons in ionized gas and atoms or molecules in neutral gas. It is **fundamentally driven by temperature gradients**: heat flows from hot regions to cold ones, acting to equilibrate temperature differences.

Proposition 4.4.1 — Fourier's Law of Conduction. To formalize this flux due to temperature gradients, we introduce **Fourier's Law of Conduction**:

The conductive heat flux $\mathbf{F}_{\rm cond}$ (energy per unit area per unit time) is proportional to the negative gradient of temperature:

$$\mathbf{F}_{\rm cond} = -\kappa \nabla T.$$

Here, κ has units of erg $\rm cm^{-1}\ s^{-1}\ K^{-1}$, and the negative sign reflects the fact that heat flows from higher to lower temperature.

We may use this to derive a **diffusion equation** for the transfer of heat. Consider a fluid element in a volume V with density ρ and temperature T. Now, the **total heat flux** due to conduction into the volume element is

$$\delta E = \int_{\partial V} -\kappa \nabla T \cdot d\mathbf{S} = -\kappa \int_V \nabla^2 T dV$$

by **divergence theorem**. Now, that energy will increase the temperature by a factor determined by the **specific heat at constant volume** c_v. Thus, $\rho c_v \delta T = \delta E$. Thus,

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c_v}\nabla^2 T = \chi \nabla^2 T,} \tag{4.10}$$

where χ is the **thermal diffusivity**.

Relevance in Astrophysics

In any system, thermal conduction is bound to play a role; however, the impact created will vary significantly. Note that χ has units $[\chi] = [L]^2\,[T]^{-1}$, which means that the **conduction timescale** for a system with a particular **length scale** L is

$$t_{\rm cond} = \frac{L^2}{\chi} = \frac{\rho c_v}{\kappa} L^2.$$

R Note that $t_{\rm cond} \sim \rho$, which means that **denser fluids are LESS efficient conductors**. This is a counterintuitive statement; clearly we think that interactions occur more frequently in denser environments. The resolution is that, while **interactions are more frequent**, there is more mass to heat and scattering prevents the conduction from occurring over large spatial scales. This is why conduction is so important in the intracluster medium of galaxy clusters which are extremely diffuse.

Magnetic Influence

One of the critical elements of the above derivation is that the **thermal diffusivity** is **isotropic** (always the same in all directions). For charged ions in **magnetic fields**, this assumption breaks down and has a dramatic influence on conduction. Because the magnetic field lines dictate the direction of the particle stream, conduction perpendicular to the field lines will be significantly reduced while it will be enhanced in the direction of the field lines.

4.4.3 Convection

Convection occurs as a fluid instability, which cases marginally hotter fluid elements to rise relative to cooler ones and relatively cooler ones to sink under specific circumstances. This can create thermal transfer on scales larger than the conduction scale.

4.4.4 Free-Free Emission (Bremsstrahlung)

Free-free emission, or **thermal bremsstrahlung**, arises when free electrons are decelerated or deflected in the Coulomb fields of ions. This acceleration leads to the emission of radiation, even though the electron remains unbound before and after the interaction. The process is especially important in ionized plasmas where temperatures are high and densities moderate to large.

The approximate volumetric emissivity (energy loss per unit volume per unit time) is:

$$\Lambda_{\rm ff} \approx 1.42 \times 10^{-40}\, Z^2\, T^{1/2}\, g_{\rm ff}(T)\, n_e\, n_p \;\; [\text{erg cm}^{-3}\ \text{s}^{-1}],$$

where:

- Z: Charge number of the ion (typically $Z = 1$ for hydrogen),
- T: Temperature of the gas [K],
- n_e: Electron number density [cm^{-3}],
- n_p: Proton (or ion) number density [cm^{-3}],
- $g_{\rm ff}(T)$: Gaunt factor (dimensionless correction factor, weakly dependent on temperature, usually ~ 11.5).

Key Properties:

- Emissivity increases with $T^{1/2}$, and especially with $n_e n_p \sim \rho^2$, so it is significant in hot, dense plasmas.
- Dominant in X-ray emission from the intracluster medium, stellar coronae, and hot winds.
- Spectral shape is broadband and roughly flat until the exponential drop at high frequencies due to the Maxwell-Boltzmann tail.

4.4.5 Recombination Emission (Free-Bound Radiation)

Recombination radiation occurs when a free electron is captured by a positive ion, transitioning to a bound state. The excess energy (kinetic + potential) is radiated away as a photon. The process dominates in regions of partial ionization (e.g., HII regions, nebulae).

The volumetric energy loss rate is given by:

$$\Lambda_{\rm fb} \approx n_e \, n_p \, kT \, \beta(H^0, T) \;\; [\text{erg cm}^{-3} \text{ s}^{-1}],$$

where:

- $\beta(H^0, T)$: A temperature-dependent recombination coefficient for hydrogen [$\text{cm}^3 \text{ s}^{-1}$], representing the rate at which energy is released per recombination,
- Other symbols as previously defined.

Notes:

- The recombination coefficient β is related to the case-A or case-B recombination rate $\alpha(T)$, which depends on whether ionizing photons are reabsorbed or escape.
- This emission often appears in emission lines (e.g., Balmer series) and contributes to the continuum at UV/optical wavelengths.

4.4.6 Collisional Excitation

Collisional excitation refers to the process where electrons collide with atoms or ions and excite their electrons to higher energy levels. The atom subsequently decays back to a lower energy state, emitting a photon. This is an important cooling process in ionized or partially ionized gases, particularly at lower temperatures ($T \sim 10^4$ K).

The approximate energy loss rate per unit volume is:

$$\Lambda_{\rm cx} \approx n_{\rm ion} \, n_e \, \chi \, \frac{8.6 \times 10^{-12}}{\sqrt{T}} \, \omega \, e^{-\chi/kT} \;\; [\text{erg cm}^{-3} \text{ s}^{-1}],$$

where:

- $n_{\rm ion}$: Number density of the relevant ion or atom,
- χ: Excitation energy of the transition [erg or eV],
- ω: A dimensionless factor depending on the collisional cross section (often includes line multiplicity, quantum corrections, etc.),
- $e^{-\chi/kT}$: Boltzmann factor capturing the energy threshold for excitation.

Important Points:

- Strong temperature sensitivity due to the exponential Boltzmann suppression at low T.
- Dominates in warm, partially ionized regions (e.g., HII regions, planetary nebulae).
- Leads to strong line cooling (e.g., [OIII], [NII], [CII] lines).

4.4.7 The form of $\dot{Q}_{\rm cool}$

For many of the mechanisms described above, $\Lambda \sim \rho^2$, which means that the cooling per unit mass is $\dot{Q}_{\rm cool} \sim \rho$. In other cases, we have seen that $\dot{Q}_{\rm cool} \sim C$. In order to incorporate most of these mechanisms heuristically, we let

$$\dot{Q}_{\rm cool} = A\rho T^{\alpha} - H.$$

5. Bernoulli's Theorem

Let's consider the **momentum equation** in its **Eulerian form**:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f}_{\text{ext}}.$$

Now, one question we might ask is if there are any dynamical **first integrals** which can be obtained from physical arguments. In general, this is not an easy undertaking; however, when the flow **is steady**, it can be shown to have a first integral in the form of **Bernoulli's Theorem**.

In order to construct this first integral, we want to effectively construct a **potential** for the entire flow which behaves analogously to that of gravitation or electrodynamics. Unfortunately, not all of the terms in the Euler equation are easily cast as gradients of scalar potentials. Consider the pressure term: Consider the pressure term:

$$-\frac{\nabla p}{\rho} = -\nabla h(\rho),$$

for some h. The equestion becomes what exactly h is. Formally, we have

$$\frac{-\nabla p}{\rho} = -\frac{1}{\rho}\frac{dp}{d\rho}\nabla\rho = -\nabla h(p).$$

If we integrate, we have

$$dh = \frac{1}{\rho}\frac{dp}{d\rho}d\rho = \frac{dp}{\rho} \implies h(p) = \int^p \frac{dp}{\rho}.$$

This h has a name: it is the **specific enthalpy of the material**.

Our equation has therefore taken the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla h + \mathbf{f}_{\text{ext}}.$$

Now, it can be shown that (vector identity),

$$(\mathbf{A}\cdot\nabla)\mathbf{A} = \frac{1}{2}\nabla(|\mathbf{A}|^2) - \mathbf{A}\times(\nabla\times\mathbf{A}).$$

If you ever need to show this, the idea is to pull in A to get the derivative of the square (i.e. chain rule) and then add a correction term. As such, we may write the second term on the LHS as

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(u^2) - \mathbf{u}\times(\nabla\times\mathbf{u}).$$

What have we gained? Here's the idea, we now have almost entirely gradient operators in the Euler equation, except for this rather odd looking curl term. Recalling from definition 1.2.2 that the **vorticity of flow** is

$$\boldsymbol{\omega} = \nabla\times\mathbf{u},$$

we may write Euler's equation in the form

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + h + \phi\right) = \mathbf{u}\times\boldsymbol{\omega}.}$$

It is from this expression that we can begin to elucidate **Bernoulli's Theorem**.

5.1 Bernoulli's Theorem

The expression above is already the **most general** form of the famous **Bernoulli's Theorem**, which we now state in full clarity:

Theorem 5.1.1 — Bernoulli's Theorem. For **any fluid flow** in which the **only external forces** are those derived from potentials, **Bernoulli's Theorem** states that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + \Phi\right) + \frac{\nabla P}{\rho} = \mathbf{u} \times \boldsymbol{\omega}. \tag{5.1}$$

Here, Φ is the external potential, ρ is the mass density, P is the pressure, and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity.

This form is exact and general, but not always directly useful in applications. The power of Bernoullis theorem lies in its **corollaries**, which arise under additional assumptions that simplify the dynamics.

5.1.1 First Corollary: Steady Barotropic Flow

Our first corollary concerns barotropic flows which are in a steady state:

Corollary 5.1.2 Consider a fluid which is in **steady-state** ($\partial_t \mathbf{u} = 0$), and which follows a **barotropic equation of state**, i.e. $P = P(\rho)$. If the only external forces are those derived from potentials (combining to form Φ), then the flow obeys

$$\mathbf{u} \cdot \nabla \mathcal{B} = 0, \tag{5.2}$$

where $\mathcal{B}$, the **Bernoulli function**, is given by

$$\mathcal{B} = \frac{1}{2}u^2 + h + \Phi,$$

and h is the **specific enthalpy**, defined by

$$h(\rho) = \int^{\rho} \frac{dP}{\rho'}.$$

An **equivalent (and more useful) statement** is

$$\boxed{\frac{D\mathcal{B}}{Dt} = 0.}$$

Proof. In steady flow the time derivative vanishes. Under the barotropic assumption,

$$\frac{\nabla P}{\rho} = \nabla h.$$

Substituting into (5.1) gives

$$\mathbf{u} \cdot \nabla \left(\tfrac{1}{2}u^2 + h + \Phi \right) = 0,$$

which is precisely (5.2). ■

Equation (5.2) shows that the Bernoulli function $\mathcal{B}$ is **constant along streamlines**. That is, as one moves with the flow, the combination of kinetic energy, enthalpy, and potential energy remains conserved. This conservation law is a cornerstone of fluid dynamics, forming the basis for practical applications ranging from aerodynamics (lift and pressure differences) to astrophysical accretion flows.

5.1.2 Second Corollary: Irrotational Flow

We may also make refinements to the Bernoulli Theorem when we have **irrotational flow**:

Corollary 5.1.3 If the flow is **irrotational**, i.e. $\boldsymbol{\omega} = 0$, then the velocity field can be written as the gradient of a scalar potential:

$$\mathbf{u} = \nabla\psi.$$

In this case, Bernoullis theorem reduces to the statement that

$$\nabla \left(\frac{\partial \psi}{\partial t} + \mathcal{B} \right) = 0.$$

Equivalently,

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}u^2 + h + \Phi = f(t),$$

where $f(t)$ is a function of time alone. This is the most general form of Bernoullis theorem for irrotational flow.

Proof. If $\boldsymbol{\omega} = 0$, then $\mathbf{u}$ is curl-free and may be expressed as the gradient of a velocity potential ψ. Substituting $\mathbf{u} = \nabla\psi$ into (5.1) yields

$$\nabla \left(\frac{\partial \psi}{\partial t} + \frac{1}{2}u^2 + h + \Phi \right) = 0,$$

which implies that the quantity in parentheses must be spatially uniform, though it may still vary in time. Hence the stated result. ■

The introduction of a velocity potential is a profound simplification: the vector equation of motion for the fluid is reduced to a scalar relation. Even in unsteady situations, the dynamics can be reformulated in terms of ψ, which greatly facilitates both analysis and computation. This scalar Bernoulli equation is the foundation of potential flow theory, allowing one to analyze flows with far less complexity than in the general rotational case.

5.1.3 Third Corollary: Steady Irrotational Flow

The final corollary worth mentioning is a refinement of the previous one in which we now also have a steady flow:

Corollary 5.1.4 If the flow is both **steady** and **irrotational**, then the Bernoulli function

$$\mathcal{B} = \frac{1}{2}u^2 + h + \Phi$$

is not only conserved along streamlines, but is in fact **constant throughout the entire flow domain and independent of time**.

Proof. In steady flow, $\partial_t \psi = 0$, so the unsteady Bernoulli relation of the previous corollary reduces immediately to

$$\frac{1}{2}u^2 + h + \Phi = \text{constant}.$$

This constant holds everywhere in the flow, not just along streamlines. ■

This is the most celebrated and practically useful form of Bernoullis theorem. It asserts that in steady, irrotational, barotropic flow under conservative forces, every fluid element shares the same Bernoulli constant. Physically, it represents the conservation of mechanical energy per unit mass the sum of kinetic energy, enthalpy, and potential energy across the entire flow field. This global constraint underlies classical results in aerodynamics (such as lift and pressure distribution), hydrodynamics (water jets, weirs, and pipe flow), and astrophysics (stellar winds and accretion flows). It is this version of Bernoullis theorem that most clearly illustrates the intimate link between fluid kinematics and conservation of energy.

Big Idea

Bernoullis Theorem is a reformulation of Eulers equations that reveals a powerful energy conservation principle in fluid flows. Its critical insights are:

- In its **general form**, the theorem is equivalent to the Euler equations, with an explicit vorticity term that prevents a simple conservation law.
- Under the assumptions of **steady, barotropic flow**, the **Bernoulli function**

  $$\mathcal{B} = \tfrac{1}{2}u^2 + h + \Phi$$

  is conserved *along streamlines*, representing the conservation of mechanical energy per unit mass (kinetic + enthalpy + potential).
- If the flow is further **irrotational**, then a velocity potential ψ exists and Bernoullis theorem reduces to a scalar relation. In this case, $\mathcal{B}$ is spatially uniform at each instant.
- In the most restrictive case of **steady, irrotational flow**, $\mathcal{B}$ is globally constant throughout the domain and independent of time the classic textbook form of Bernoullis law.

Takeaway: The hierarchy of Bernoulli results reflects progressively stronger assumptions: from a general Eulerian identity, to conservation along streamlines, to global constancy in potential flows. This framework underpins the use of Bernoullis equation in engineering, aerodynamics, and astrophysics.

5.2 Vorticity and Kelvin's Circulation Theorem

Some more is worth saying on the subject of **vorticity**. In a general flow, where the requirements of irrotational flow and steady flow have been removed, we still have a **momentum equation** in the form

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \mathcal{B} + \mathbf{u} \times \boldsymbol{\omega}.$$

From this result, we may arrive at the very useful **Helmholtz's Theorem**:

Theorem 5.2.1 — Helmholtz's Theorem. For any fluid which is **irrotational** at a time t_0, then for any prior or later time t, the fluid must remain irrotational unless viscous forces are at play.

Proof. If we take the curl of both sides of the equation above, we find

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\nabla \times \nabla \mathcal{B} + \nabla \times (\mathbf{u} \times \boldsymbol{\omega}).$$

Relying on the fact that the curl of a gradient is 0, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}).$$

As such, if $\boldsymbol{\omega} = 0$, then it cannot evolve in time. ■

We can actually take this concept even further. Formally, consider a **simple-closed contour (loop)** in the flow ($\partial S(t)$). *You can think if this as a set of fluid elements at time t_0 which form a contour and then evolve with the flow from there, defining the contour in time.* Now, that contour will (of course), define a surface and the **vorticity flux** across that contour is

$$F = \int_{S(t)} \boldsymbol{\omega} \cdot d\mathbf{S}.$$

We might ask how this evolves as the fluid moves with time. The **Lagrangian Derivative** will be required to incorporate the change in the vorticity, but also the change in the contour as well. We can define an equivalent quantity to F: the **circulation** around the contour. By Stoke's theorem,

$$\Gamma = \int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{u} \cdot d\mathbf{l}.$$

Now, let's parameterize the **lagrangian configuration** of the boundary as $\alpha \in [0,1]$ such that

$$d\mathbf{l} = \partial_\alpha \mathbf{X} d\alpha.$$

the change in the circulation with time is then

$$\frac{D\Gamma}{Dt} = \frac{d}{dt}\int_0^1 d\alpha \mathbf{u}(\mathbf{X},t)\cdot\partial_\alpha\mathbf{X}.$$

Pulling the time derivative in, we have

$$\frac{D\Gamma}{Dt} = \int_0^1 \frac{D\mathbf{u}}{Dt}\cdot\partial_\alpha\mathbf{X} + \mathbf{u}\cdot\partial_\alpha\frac{D\mathbf{X}}{Dt}\,d\alpha$$

Now, $D\mathbf{X}/Dt = \mathbf{u}(\mathbf{X},t)$, so that second term has the form

$$\int_0^1 (\mathbf{u}\cdot\partial_\alpha)\mathbf{u}\,d\alpha = \int_0^1 (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\partial_\alpha\mathbf{X}\,d\alpha$$

Now, if we expand the $D\mathbf{u}/Dt$, term as well, it cancels with this term above and we get

$$\frac{D\Gamma}{Dt} = \oint \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{l}.$$

If we now apply the **Euler Equation** in the special case where **Bernoulli's theorem holds**, then this RHS is a potential and therefore has no circulation. This implies **Kelvin's Circulation Theorem**, which claims that

$$\frac{D\Gamma}{Dt} = 0.$$

5.3 Vorticity Transport and Stretching

Read / write about this a bit

5.4 Aside: Potential Flow

A worthwhile note at this stage is the following: if we have an **irrotational fluid**, then there exists some ψ such that

$$\mathbf{u}(\mathbf{x},t) = \nabla\psi(\mathbf{x},t).$$

This occurs because $\mathbf{u}$ has no curl if it is irrotational. Additionally, if we have a **incompressible fluid**, then

$$\nabla\cdot\mathbf{u} = 0 \implies \nabla^2\psi = 0.$$

As such, these flow problems are solved via the **Laplace equation**.

5.5 DeLaval's Nozzle

Consider the following very basic problem:

A pipe of cross sectional area $A(z)$ carries a steady flow with variables $u(z)$, $\rho(z)$, and $p(z)$. Determine the relationship between each of these and the cross sectional area.

Now, in this scenario, we will **ignore gravity** and instead treat the simpler problem of a gas in a pipe and its relevant velocity and pressure. We'll likewise assume a **barotropic equation of state** as we have done in many previous scenarios.

Since the flow is **steady**, we cannot allow mass to build up anywhere. This gives us our first piece of information. We know then that

$$\rho u A = \text{constant}.$$

Now, the Euler equation for momentum tells us that

$$\frac{D\mathbf{u}}{Dt} = (u \cdot \nabla)u = -\frac{dp}{d\rho} d\log\rho.$$

From mass conservation, we have

$$\log\rho + \log u + \log A = C \implies d\log\rho = -d\log u - d\log A.$$

We can thus write

$$\frac{-dp}{d\rho} d\log\rho = \frac{dp}{d\rho}\left[d\log u + d\log A\right] = c_s^2[d\log u + d\log A].$$

Now, this is where we make our **tricky simplification**: we assume that the flow is **irrotational** (a valid assumption), and so

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u}\times\boldsymbol{\omega} = \frac{1}{2}\nabla u^2.$$

This can be written as $u\nabla u = u^2\nabla\log u$, so

$$\boxed{(u^2 - c_s^2)d\log u = c_s^2 d\log A}$$

R You can play around with this mentally quite a bit. Let's consider some scenarios where u starts out subsonic so that we have $-d\log u \sim d\log A$.

- As we **contract the nozzle**, we **increase velocity**, but it requires more and more contraction as we get closer to c_s. we **cannot make a sonic transition** until the nozzle reverts.

If we are **supersonic**, then we instead need $d\log A \sim d\log u$, so the nozzle expands to move our fluid faster. This is the idea behind which rocket engines build their jets. We start with a thermal energy rich fluid with low velocity. As it moves down the nozzle, it speeds up until the center, when the nozzle starts expanding again, the speed goes super-sonic and a lot of that thermal energy is converted to kinetic energy. This is a **very effective way to convert thermal to mechanical energy**.

The De Laval Nozzle can be used to partially describe the nature of astrophysical jets; however, other (magnetohydrodynamical) effects are relevant in these scenarios.

5.6 Bondi Accretion

Bondi accretion is the simplest classical model of accretion in a spherically symmetric system. It relies on several significant assumptions and derives the relevant hydrodynamic behavior of the system. In this section, we will perform the derivation and cite the relevant limitations.

5.6.1 The Intuitive Picture

The full hydrodynamic derivation of Bondi accretion is rigorous but not something one needs to memorize. What is most useful to retain is the simple physical intuition that leads directly to the correct scaling for the accretion rate.

The key concept is the existence of an **accretion radius**, $r_{\rm acc}$, where the gravitational binding energy of the accretor balances the thermal energy of the gas:

$$\frac{GM}{r_{\rm acc}} \sim c_s^2(\infty).$$

Inside this radius, gravity dominates over pressure support, and gas is effectively captured. Thus $r_{\rm acc} \sim GM/c_s^2(\infty)$ defines the size of the accretor's sphere of influence.

Now, how much material is drawn in? Imagine that gas within this radius is funneled inward at roughly the sound speed, $c_s(\infty)$, since this is the characteristic velocity scale of the medium. The *feeding rate* of mass across the surface of the capture sphere is then

$$\dot{M} \sim \rho_\infty \, v \, A \qquad \sim \qquad \rho_\infty \, c_s(\infty) \, \pi r_{\rm acc}^2.$$

Substituting the scaling for $r_{\rm acc}$,

$$\dot{M} \;\sim\; \pi \rho_\infty \left(\frac{GM}{c_s^2(\infty)} \right)^2 c_s(\infty).$$

Thus we immediately arrive at the Bondi scaling

$$\boxed{\dot{M} \;\propto\; \frac{(GM)^2 \rho_\infty}{c_s^3(\infty)}.}$$

This heuristic argument is not exactit ignores the subtleties of transonic flow and the precise dependence on the polytropic index γbut it captures the correct dependence on M, ρ_∞, and c_s. The rigorous hydrodynamic treatment merely supplies the order-unity prefactor. The essential physics is therefore straightforward: accretion is controlled by the size of the gravitational capture region and by the rate at which gas can flow into it at the ambient sound speed.

Assumptions

In Bondi accretion, we make the following simplifying assumptions:

1. **Spherical symmetry:** The entire flow is treated as spherically symmetric and the accretor is at rest relative to the ambient material. *This plays out mathematically immediately.*

2. **Steady State**: The nature of the flow does not change over time. Formally, this means that any of the field variables ψ is independent of time. *This is a trickier requirement as it allows us to define a constant accretion rate; however, we know this to be not in keeping with physical systems.*
3. **Polytropic Equation of State**: We assume a *barotropic equation of state* following the form of a polytrope

$$P = \kappa\rho^{\gamma}.$$

In the limiting cases, we have either adiabatic flows (optically thin) or isothermal (optically thick).
4. **Gravity**: Is assumed to be fully Newtonian and the accretor is treated as a point mass.
5. **No Additional Forces**: The only relevant force is that of gravity. MHD effects are ignored.

Derivation

Formally, we have 3 equations: the **continuity equation**, the **Euler equation**, and the **equation of state**. In this scenario, continuity provides that

$$\underbrace{\frac{\partial\rho}{\partial t}}_{=0\text{ (assumpt. 2)}} + \nabla\cdot(\rho\mathbf{u}) = 0 \implies \frac{1}{r^2}\partial_r[r^2\rho\mathbf{u}] = 0.$$

We therefore find that

$$r^2\rho\mathbf{u} = \text{Constant}.$$

This is a very useful integral of the motion because the accretion rate is

$$\dot{M} = -4\pi r^2\rho u = \text{Constant}.$$

This is deducible from the fact that we have steady flow and therefore cannot collect mass in shells.

We also have the **Euler Equation** in the form

$$u\frac{du}{dr} + \frac{1}{\rho}\frac{dP}{dr} + \frac{GM}{r^2} = 0.$$

Using the polytropic equation of state,

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2\frac{d\rho}{dr}.$$

Remember that c_s^2 is a function of radius. From the continuity equation,

$$\frac{1}{r^2}\partial_r(\rho r^2 u) = 0 \implies \underbrace{\rho\frac{1}{r^2}\partial(r^2u) + u\partial_r\rho}_{\text{product rule}} = 0 \implies \partial_r\log\rho = -\frac{1}{r^2u}\partial_r u r^2,$$

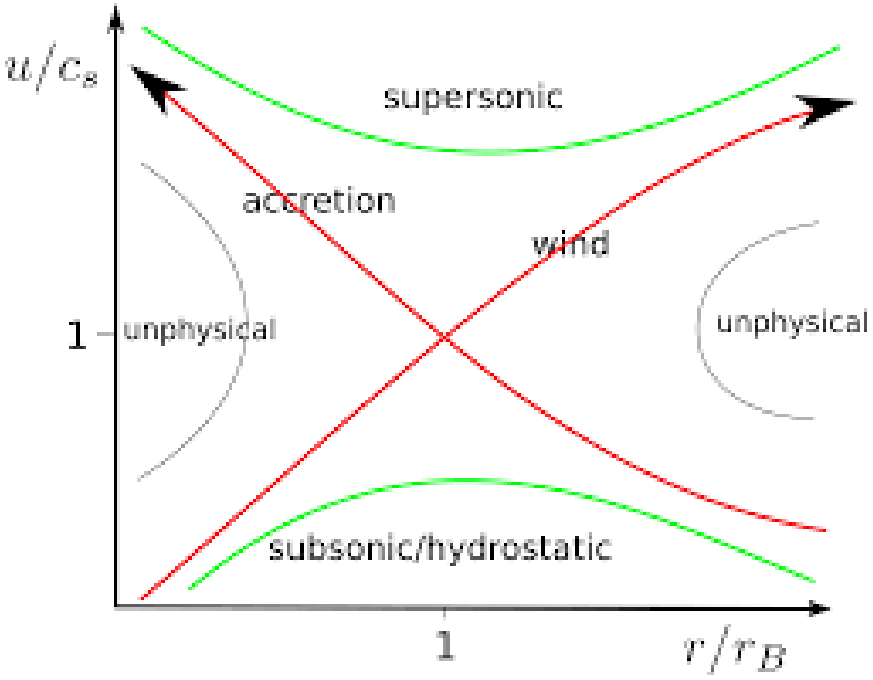


Figure 5.1: The parameter space of Bondi-accretion solutions. The sonic point r_b is shown on the x-axis and the velocity on the y axis.

so (*substitute* $\partial_r \log \rho$ *and then expand out the prod. rule*),

$$u\frac{du}{dr} - \frac{c_s^2}{ur^2}\frac{du}{dr} + \frac{GM}{r^2} = 0.$$

If we perform some rearrangements, we find the critical equation which will consume our discussion for the rest of the section:

$$\boxed{\frac{1}{2}\left(1 - \frac{c_s^2}{u^2}\right)\frac{du^2}{dr} = -\frac{GM}{r^2}\left[1 - \frac{2c_s^2 r}{GM}\right].} \tag{5.3}$$

The Sonic Point

It is not immediately clear why we should have gone to all the work of building out this complicated ODE for ourselves; however, we can see that there are several very interesting features. The most important of these is that equation (5.4) is **singular** at

$$\boxed{r_s = \frac{GM}{2c_s^2}.} \tag{5.4}$$

Remember that c_s *is still a function of the radius. This means that this is* ***implicit****.* This is the so-called **sonic point** of the flow: *if* the flow is going to make a transition to or from **sub-sonic** to **super-sonic**, it *must occur at* r_s*.* This means that there are 4 important regimes to consider:

1. **Transitioning Solution**: If we enforce that $u = c_s$ at the sonic radius, then the solution is entirely determined by the choice of behavior as $r \to \infty$ or by the choice of behavior as $r \to 0$. If we let $u \to 0$ at ∞, then we obtain **accreting flows** featuring a transition point, and if we permit $u \to 0$ as $r \to 0$, then we obtain **wind flows** with transition points.
2. **Non-Transitioning Solutions**: If a solution is not going to have $u = c_s$, then one *must let* $du^2/dr = 0$ at the sonic radius. In this case, the entire solution is fixed either by the behavior at either asymptote or by the behavior (the velocity) at the sonic point.

Bernoulli Flow

We have now clarified the general behavior of the ODE we wish to solve and identified the relevant regimes. Most importantly, we are now able to recognize that uniqueness of our solution can be guaranteed by specifying the behavior both at the critical point and at ∞. Let us now fully solve the problem. To do so, we will apply **Bernoulli's Theorem**:

$$u\frac{du}{dr} + \nabla(h + \phi) = 0 \implies \frac{1}{2}u^2 + h + \phi = 0.$$

For a **barotropic equation of state**, the specific enthalpy is

$$h = \int \frac{dP}{\rho} = \int \frac{dP}{d\rho}\frac{1}{\rho}d\rho = \frac{K\gamma}{\gamma - 1}\rho^{\gamma-1} = \frac{c_s^2}{\gamma - 1}.$$

Thus,

$$\frac{u^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \text{Constant}. \tag{5.5}$$

In the isothermal case, we actually need to have a logarithm here instead of $\gamma - 1$.

Now, for **accreting flows**, we have $u(\infty) = 0$. Thus,

$$c_s^2(\infty) = C(\gamma - 1) \implies C = \frac{c_s^2(\infty)}{\gamma - 1}.$$

Additionally, the sonic point requires that

$$c_s^2(r_s) = \frac{GM}{2r_s},$$

so at r_s, we have

$$\frac{c_s^2(r_s)}{2} + \frac{c_s^2(r_s)}{\gamma - 1} - 2c_s^2(r_s) = \frac{c_s^2(\infty)}{\gamma - 1}.$$

So

$$\boxed{c_s(r_s) = c_s(\infty)\left(\frac{2}{5 - 3\gamma}\right)^{1/2}} \tag{5.6}$$

Once again, we see that $\gamma = 5/3$ is not permitted! In practical scenarios, we are never really ideal, so this is fine, but nonetheless, it is worth noting. The mass accretion rate is constant at all radii, so we can evaluate it at the sonic point and find

$$\boxed{\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)} \left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}}. \tag{5.7}$$

The Accretion Radius

The Bondi solution motivates the introduction of a characteristic length scale, the **accretion radius**, defined as

$$r_{\rm acc} \equiv \frac{2GM}{c_s^2(\infty)}. \tag{5.8}$$

This can be understood from a simple energetic argument. At radius r, the **gravitational binding energy per unit mass** is

$$E_{\rm grav} \sim \frac{GM}{r},$$

while the **thermal energy per unit mass** of the gas is set by the sound speed,

$$E_{\rm th} \sim c_s^2(\infty).$$

The radius $r_{\rm acc}$ is defined as the point where these two energy scales balance: inside this radius, gravitational attraction dominates over thermal motions, so gas is gravitationally captured by the accretor. Outside this radius, pressure forces can support the gas against collapse. Thus, $r_{\rm acc}$ plays the role of an effective "sphere of influence" for accretion.

R Note that $r_{\rm acc}$ is distinct from the precise sonic radius r_s, which depends on the local sound speed $c_s(r_s)$. The accretion radius is defined in terms of the *asymptotic* sound speed at infinity, and provides a more intuitive, order-of-magnitude measure of the capture region.

Free-Fall Behavior Beyond the Sonic Point

Once the gas passes through the sonic point, the flow is **supersonic**. In this regime, pressure forces are negligible compared to inertia and gravity: the gas effectively undergoes free fall. This allows us to extract the asymptotic scaling of velocity, density, and temperature in the inner region.

Velocity:

In free fall onto a point mass, the velocity is set by the gravitational potential:

$$u(r) \sim \left(\frac{2GM}{r}\right)^{1/2}.$$

Density:

The accretion rate is constant at all radii,

$$\dot{M} = 4\pi r^2 \rho u.$$

Substituting the free-fall velocity,

$$\rho(r) \sim \frac{\dot{M}}{4\pi r^2 u(r)} \propto r^{-3/2}.$$

Temperature:
For a polytropic gas,

$$T \propto \frac{P}{\rho} \propto \rho^{\gamma-1}.$$

Thus, in the inner free-fall region,

$$T(r) \ \propto \ r^{-3(\gamma-1)/2}.$$

For example:

- Isothermal case ($\gamma = 1$): $T(r) = \text{const}$.
- Adiabatic monoatomic gas ($\gamma = 5/3$): $T(r) \propto r^{-1}$.

Summary and Key Formulae

Bondi accretion provides the simplest classical model for spherically symmetric accretion onto a compact object. While highly idealized, it illustrates several key physical principles:

- The flow is uniquely determined by the requirement that it pass smoothly through the **sonic point**. This makes the solution transonic, subsonic at infinity and supersonic near the accretor.
- The **accretion radius**

  $$r_{\rm acc} = \frac{2GM}{c_s^2(\infty)}$$

  defines the natural scale inside which gravity dominates over thermal pressure. Gas at $r \lesssim r_{\rm acc}$ is gravitationally captured.
- The **Bondi accretion rate** can be expressed either in terms of $r_{\rm acc}$ or directly in terms of asymptotic conditions:

  $$\dot{M}_{\rm Bondi} \ \sim \ \pi r_{\rm acc}^2 \rho_\infty c_s(\infty), \qquad \dot{M}_{\rm Bondi} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s(\infty)^3} \left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)}.$$

  The exact prefactor depends on the adiabatic index γ, but the scaling is robust.
- Inside the sonic point, the flow is effectively in free fall, with the following power-law scalings:

  $$u(r) \propto r^{-1/2}, \qquad \rho(r) \propto r^{-3/2}, \qquad T(r) \propto r^{-3(\gamma-1)/2}.$$

  For an adiabatic monoatomic gas ($\gamma = 5/3$), this gives $T(r) \propto r^{-1}$.
- Order-of-magnitude estimates show that for compact objects, the rates are very small compared to what is observable:

  $$\dot{M}_{\rm Bondi} \sim 1.4 \times 10^{11} \ \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\rho_\infty}{1\,{\rm cm}^{-3}\ m_p}\right) \left(\frac{10\,{\rm km/s}}{c_s(\infty)}\right)^3 \ {\rm g\,s^{-1}}.$$

  For example:
  - White Dwarf ($M \sim 1M_\odot$, $R \sim 10^9\,$cm): $\dot{M} \sim 10^{12}\,{\rm g\,s^{-1}}$.
  - Neutron Star ($M \sim 1.4M_\odot$, $R \sim 10^6\,$cm): $\dot{M} \sim 10^{13}\,{\rm g\,s^{-1}}$.

  Even though compact objects have deep gravitational potentials, the sparse interstellar medium is simply too dilute: Bondi accretion in realistic astrophysical settings is far below detectable levels.

Big Idea

Bondi Accretion: Must-Remember Formulae

$$r_{\rm acc} = \frac{2GM}{c_s^2(\infty)}$$

$$\dot{M}_{\rm Bondi} \sim \pi r_{\rm acc}^2 \rho_\infty c_s(\infty) \quad \propto \frac{(GM)^2 \rho_\infty}{c_s^3(\infty)}$$

$$\rho(r) \propto r^{-3/2}, \qquad T(r) \propto r^{-3(\gamma-1)/2}, \qquad u(r) \propto r^{-1/2}$$

$$\dot{M}_{\rm WD} \sim 10^{12} \left(\frac{M}{M_\odot}\right)^2 \ {\rm g\,s^{-1}}, \qquad \dot{M}_{\rm NS} \sim 10^{13} \left(\frac{M}{M_\odot}\right)^2 \ {\rm g\,s^{-1}}$$

Takeaway: Bondi accretion sets the baseline scale for spherical capture from a uniform medium, but the resulting accretion rates are far too small to be astrophysically significant in most environments.

6. Viscosity

In this chapter, we'll dive more cleanly into the world of viscous flows and their relevance to astrophysical fluid dynamics. We begin from where we left off on the topic in the first few chapters. Here, we'll try to touch more on the astrophysical relevance of these sorts of flows and derive some important results.

6.1 Foundations of Viscous Flow

Before diving into new details, let's review the foundational information previously discussed in connection with the fluid equations and use them to build some additional sophistication.

Review of Previous Results

We have seen that viscosity can be formed into a stress tensor τ_{ij}

$$\boldsymbol{\tau} = \mu\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right] + \mathbf{I}\left(\zeta - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u},$$

where μ is the **shear viscosity** and ζ is the **bulk viscosity**. From this, we transformed the Euler Equation into the **Navier-Stokes Equation** in the form (see equation 2.12)

$$\boxed{\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla P + \mu\nabla^2\mathbf{u} + \left(\zeta + \tfrac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F}_{\text{ext}}.} \tag{6.1}$$

In the **incompressible limit**, we can eliminate the term involving the **bulk viscosity** and arrive at the incompressible NS-equation:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla P + \mu\nabla^2\mathbf{u} + \mathbf{F}_{\text{ext}}.$$

It will be the behavior of the above two equations that we focus on in this chapter. In this chapter, we will attempt to build a more concrete understanding of the implications

of viscosity on astrophysical flows. Now, in **astrophysical fluids**, the **bulk viscosity is generally negligible** and so we can ignore it in most scenarios. From equation (2.13), we obtain the standard analog to the Euler Equation that we will use:

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{u} + \frac{1}{3}\mu\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F}_{\text{ext}}.} \tag{6.2}$$

We will find it useful to introduce the **kinematic viscosity parameter** $\nu = \mu/\rho$ so that this equation takes the form

$$\boxed{\frac{D\mathbf{u}}{Dt} = -\frac{\nabla P}{\rho} + \nu\nabla^2\mathbf{u} + \frac{1}{3}\nu\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F}_{\text{ext}}.} \tag{6.3}$$

Let us now use these foundational expressions to derive more information about the nature of the flow.

6.1.1 Vorticity Transport in Viscous Flow

As we did previously for the Euler Equation, we will now take the curl of the above equation and thereby study the behavior of the **fluid's vorticity throughout the flow**. Clearly

$$\nabla\times\frac{D\mathbf{u}}{Dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}),$$

so

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = \nabla\times\left[-\frac{\nabla P}{\rho} + \nu\nabla^2\mathbf{u} + \frac{1}{3}\nu\nabla(\nabla\cdot\mathbf{u}) + \mathbf{F}_{\text{ext}}\right].$$

The curl of a gradient is zero, so the $\nu\nabla(\nabla\cdot\mathbf{u})$ term must be zero. Likewise, the curl term acting on the pressure gradient is

$$\nabla\times\frac{\nabla P}{\rho} = -\frac{1}{\rho^2}\nabla\rho\times\nabla P = 0.$$

*We have made two simplifications under the hood here. First, the product rule containing $\nabla\times\nabla P = 0$ by definition. Additionally, $\nabla\rho\times\nabla P$ is zero for a **barotropic fluid** since they share level surfaces.* We therefore have the following expression **for the evolution of vorticity**:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = -\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) + \nu\nabla^2\boldsymbol{\omega} + \nabla\times\mathbf{F}_{\text{ext}}.$$

Now, if we recall from our study of vorticity in previous chapters,

$$\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = \nabla\times\left[\frac{1}{2}\nabla(\mathbf{u}^2) - \mathbf{u}\times(\nabla\times\mathbf{u})\right] = -\nabla\times\mathbf{u}\times\boldsymbol{\omega},$$

so if we have only gravitational forces (or any other conservative force), then the external force term disappears and we have,

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega}.$$

Let's take a minute to understand this. Recall our discussion of Kelvin's Vorticity Theorem that, in **inviscid flows**,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}),$$

And so we see that there is a new, Laplace-like (diffusion) element to our vorticity evolution. First, we see that **Kelvin's Circulation Theorem is preserved** in the sense that if the flow begins with $\boldsymbol{\omega} = 0$, then no vorticity will be generated in time. The caveat is that this only holds for **barotropic fluids** with suitable forces (e.g. gravity); in more general cases, baroclinic effects ($\nabla\rho \times \nabla P \neq 0$) can generate vorticity.

Second, viscosity introduces an entirely new effect: a **diffusion term** $\nu\nabla^2\boldsymbol{\omega}$. In an inviscid fluid, a compact region of vorticity (say a vortex ring) is simply advected and deformed by the flow without spreading. By contrast, in a viscous fluid, momentum exchange across streamlines causes the vorticity to diffuse outward, weakening the core and smearing rotational structures into the surrounding fluid.

Big Idea

In inviscid (Euler) flows, vorticity is conserved and carried along with the fluid, as expressed by Kelvin's Circulation Theorem. In viscous (Navier–Stokes) flows, an additional diffusion term appears in the vorticity equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega}.$$

- If the flow begins irrotational ($\boldsymbol{\omega} = 0$), it remains so in barotropic fluids with conservative body forces.
- Viscosity causes vorticity to *diffuse*, smoothing sharp vortical structures in the same way that heat or dye diffuses in a medium.

6.1.2 Energy Transport in Viscous Flows

We are now required, once again, to redo some of our earlier analysis to account for the presence of viscosity. In this case, we will discuss in more detail the energy equation, the first law of thermodynamics, and the role that viscosity plays in these processes. As we will discover, viscosity is a **dissipative process**, which serves to heat up the fluid.

The Internal Energy

Recall that, for a thermodynamic system, the change in internal energy per unit mass is

$$d\epsilon = dq + W,$$

where W is the work done on the fluid element. In the isotropic case of an ideal pressure, this reduces to the familiar PdV term. However, when the dynamics are governed by a general stress tensor, the analysis requires more care.

Let's begin by imagining a fluid element which is (due to stresses) **expanding or contracting**. We know that, for a region of the surface $d\mathbf{A}$, the force on that element will

be

$$d\mathbf{F} = \boldsymbol{\sigma}\, d\mathbf{A} = \sigma_{ij}\, dA^j,$$

where $\boldsymbol{\sigma}$ is the stress tensor. In some period Δt, that surface will be displaced $\mathbf{u}\,\Delta t$ under the imposed force. Thus, **the work required to obtain that deformation** is

$$dW = d\mathbf{F} \cdot d\boldsymbol{\ell} = \sigma_{ij} u^i\, dA^j\ \Delta t.$$

Integrating over the **entire surface of the element**, we find the total rate of work

$$\frac{\Delta W}{\Delta t} = \oint_{\partial V} \sigma_{ij} u^i\, dA^j.$$

Applying the divergence theorem converts this surface integral into a volume integral:

$$\frac{\Delta W}{\Delta t} = \int_V \left(\sigma_{ij} u^i\right)_{;j} dV = \int_V u^i \sigma_{ij;j} + \sigma_{ij} u^i_{;j}\, dV.$$

Now, these two terms each have **different implications for the flow**:

1. The **first term** $u^i \sigma_{ij;j}$ corresponds to work done to change the **bulk flow velocity**. To see this, notice that (from the Navier-Stokes equation),

   $$\mathbf{u}\rho \cdot \frac{D\mathbf{u}}{Dt} = \mathbf{u} \cdot \nabla \cdot \boldsymbol{\sigma} + \mathbf{u} \cdot \rho \mathbf{F}_{\text{ext}}.$$

   The first term is $D(u^2)/Dt$, which is effectively the bulk kinetic energy and the resulting expression is

   $$\rho \frac{D(u^2)}{dt} \sim \mathbf{u} \cdot \nabla \boldsymbol{\sigma},$$

   which is exactly the term we've just been discussing. As such, we determine that the work from this term contributes to the **bulk energy** of the fluid.
2. The **second term** $\sigma_{ij} u^i_{;j}$ corresponds to the **heat generation** of the process. This is the energy we actually dissipate.

We therefore identify the internal contribution to the work as

$$\dot{w}_{\text{int}} = \sigma_{ij} u^i_{;j} = \boldsymbol{\sigma} : \nabla \mathbf{u}.$$

Now recall that the stress tensor may be decomposed into an isotropic pressure part and a viscous part:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}.$$

Contracting with the velocity gradient yields

$$\sigma_{ij} u^i_{;j} = -P \nabla \cdot \mathbf{u} + \tau_{ij} u^i_{;j}.$$

The first term $-P\nabla\cdot\mathbf{u}$ is the familiar compressional PdV work, while the second term $\tau_{ij}u^i_{;j} = \boldsymbol{\tau}:\nabla\mathbf{u}$ corresponds to **viscous dissipation**, i.e. the irreversible conversion of bulk kinetic energy into heat. To complete this discussion, we'll write this as the **volume rate of dissipation**:

$$\boxed{D(\mathbf{r}) = \nabla\mathbf{u}\boldsymbol{\tau} = u^i_{;j}\tau_{ij}.} \tag{6.4}$$

Finally, including radiative or other thermal losses at a rate $-\rho\dot{q}_{\rm cool}$, the first law of thermodynamics for a viscous fluid element takes the form

$$\boxed{\rho\frac{D\epsilon}{Dt} = -P\nabla\cdot\mathbf{u} - \rho\dot{q}_{\rm cool} + \boldsymbol{\tau}:\nabla\mathbf{u}.} \tag{6.5}$$

Dividing through by the density and using the continuity equation to remove the $\nabla\cdot\mathbf{u}$, we obtain the specific form

$$\boxed{\frac{D\epsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} - \dot{q}_{\rm cool} + \frac{1}{\rho}\boldsymbol{\tau}:\nabla\mathbf{u}.} \tag{6.6}$$

The two reversible contributions to the internal energy are compressional heating and radiative cooling, while the final term represents the *viscous heating rate*. For a Newtonian fluid, $\boldsymbol{\tau}:\nabla\mathbf{u}$ is a positive-definite quadratic function of velocity gradients, ensuring that viscous dissipation always increases the internal energy of the system.

The Energy Equation

Let's once again visit the **Energy Equation** of the previous chapters and, this time, fold in the viscosity to get a picture of how this influences the dynamics. Consider a fluid which has energy in its thermal motion, in its bulk motion, and in gravitational potential. The **total energy** is

$$\mathcal{E} = \rho\left(\frac{1}{2}u^2 + \Phi + \epsilon\right).$$

The time evolution of this quantity is of considerable interest. We may take the Lagrangian time derivative to find

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\nabla\cdot\mathbf{u} + \rho\left[\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \frac{D\Phi}{Dt} + \frac{D\epsilon}{Dt}\right]$$

It is here that things will differ from the derivative already familiar to us. We now need to use not the Euler Equation, but the **Navier Stokes Equation**. In order to keep things reasonably tame notationally, we'll use the convective form:

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla P}{\rho} + \frac{1}{\rho}\nabla\cdot\tau - \nabla\Phi.$$

Thus,

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\nabla\cdot\mathbf{u} + \rho\left[\mathbf{u}\cdot\left(\frac{-\nabla P}{\rho} - \nabla\Phi + \frac{\nabla\cdot\tau}{\rho}\right) + \frac{\partial\Phi}{\partial t} + \mathbf{u}\cdot\nabla\Phi + \frac{D\epsilon}{Dt}\right]$$

Clearly, the gravitational terms cancel out and we have

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\nabla\cdot\mathbf{u} + \rho\mathbf{u}\cdot(\nabla\cdot\tau - \nabla P) + \rho\frac{\partial\Phi}{\partial t} + \rho\frac{D\epsilon}{Dt}.$$

Substituting in our expression for $D\epsilon/Dt$, we have

$$\frac{D\mathcal{E}}{Dt} = -\mathcal{E}\nabla\cdot\mathbf{u} + \rho\mathbf{u}\cdot(\nabla\cdot\tau - \nabla P) + \rho\frac{\partial\Phi}{\partial t} - P\rho\nabla\cdot\mathbf{u} - \rho\dot{q}_{\text{cool}} + \rho\boldsymbol{\tau}:\nabla\mathbf{u}.$$

We see $-\mathbf{u}\cdot\nabla P - P\cdot\nabla\mathbf{u}$ and identify it as $\nabla\cdot(P\mathbf{u})$. Thus,

$$\frac{D\mathcal{E}}{Dt} + \mathcal{E}\nabla\cdot\mathbf{u} + \nabla\cdot P\mathbf{u} = \mathbf{u}\rho\cdot(\nabla\cdot\tau) + \rho\frac{\partial\Phi}{\partial t} - \rho\dot{q}_{\text{cool}} + \rho\boldsymbol{\tau}:\nabla\mathbf{u}.$$

Finally, we recall that

$$\rho\left[u^i\tau_{ij;j} + u^i_{;j}\tau_{ij}\right] = \rho\nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{u}),$$

so we obtain our **Lagrangian Energy Equation**:

$$\boxed{\frac{D\mathcal{E}}{Dt} + \mathcal{E}\nabla\cdot\mathbf{u} + \nabla\cdot(P\mathbf{u}) - \nabla\cdot(\boldsymbol{\tau}\mathbf{u}) = \rho\frac{\partial\Phi}{\partial t} - \rho\dot{q}_{\text{cool}}.} \tag{6.7}$$

In the **Eulerian Form**, we have

$$\boxed{\frac{\partial\mathcal{E}}{\partial t} + \nabla\cdot[(\mathcal{E}+P)\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\tau}] = \rho\frac{\partial\Phi}{\partial t} - \rho\dot{q}_{\text{cool}}.} \tag{6.8}$$

Big Idea

Viscous Energy Dissipation: Key Takeaways

- In a viscous fluid, the *internal energy equation* gains an additional source term:

  $$\frac{D\epsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} - \dot{q}_{\text{cool}} + \frac{1}{\rho}\boldsymbol{\tau}:\nabla\mathbf{u}.$$

  The final term is always positive and represents **viscous heating**.
- The *total energy equation* acquires a new flux contribution:

  $$\nabla\cdot\left[(\mathcal{E}+P)\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\tau}\right],$$

  describing transport of energy by viscous stresses in addition to pressure and advection.
- Physically: pressure gradients do reversible PdV work, while viscosity irreversibly converts bulk kinetic energy into heat and redistributes energy through stresses.

- Conservation remains intact: total energy is conserved, but the partition between kinetic and thermal energy shifts irreversibly due to viscous dissipation.

6.2 The Origin and Relevance of Viscosity

Up to this point, we have treated viscosity as a known physical property entering the Navier–Stokes equations through the stress tensor. However, the true origin of viscosity—and the scales at which it becomes dynamically relevant—differ profoundly between terrestrial and astrophysical fluids. In this section, we will outline the physical basis of viscosity, identify when it matters, and connect these ideas to the much broader problem of **momentum transport** in astrophysical systems.

6.2.1 The Reynolds Number

It is worthwhile and interesting to ask the following question: **When is viscosity actually relevant?** To answer this question, let's look at the NavierStokes equation again in the **incompressible limit**:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\nabla^2\mathbf{u} = -\nabla P + \mathbf{F}_{\text{ext}}.$$

The left-hand side may contain regions where the **advective (inertial)** term dominates and regions where the **viscous** term dominates. To quantify this, we can take the ratio of their characteristic magnitudes:

$$N = \frac{\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})}{\mu\nabla^2\mathbf{u}}.$$

If we consider a system with a characteristic length scale L and velocity scale U, we can estimate this ratio by dimensional analysis:

$$N \sim \frac{\rho U^2/L}{\mu U/L^2} \sim \frac{\rho UL}{\mu} \equiv \frac{UL}{\nu},$$

where $\nu = \mu/\rho$ is the **kinematic viscosity**. This dimensionless number controls the relative importance of inertial transport to viscous diffusion.

Definition 6.2.1 — Reynolds Number. The **Reynolds number** (**Re**) is the ratio of inertial forces to viscous forces in a flow:

$$\boxed{\text{Re} = \frac{UL}{\nu} = \frac{\rho UL}{\mu}.}$$

It characterizes the degree to which viscosity influences the dynamics of a fluid.

When $\text{Re} \ll 1$, viscous stresses dominate and the flow is **laminar**: smooth, ordered, and largely reversible. Each parcel of fluid moves predictably, and neighboring streamlines

remain coherent. This regime is sometimes called ***Stokes flow*** and is common for highly viscous or slow-moving fluids.

Conversely, when $\mathrm{Re} \gg 1$, inertial terms dominate. The flow becomes **turbulent**: small perturbations grow, eddies develop, and energy cascades to smaller scales. In this regime, viscosity becomes dynamically negligible on large scales, though it still plays a crucial role at the smallest (dissipation) scales by converting kinetic energy into heat.

We can equivalently interpret the Reynolds number as a ratio of **timescales**:

$$\mathrm{Re} \;=\; \frac{t_{\mathrm{visc}}}{t_{\mathrm{adv}}} \quad \text{where} \quad t_{\mathrm{visc}} \sim \frac{L^2}{\nu}, \qquad t_{\mathrm{adv}} \sim \frac{L}{U}.$$

Thus, Re tells us whether a parcel of fluid moves across a region faster than viscosity can diffuse momentum through it.

Big Idea

Physical Meaning of the Reynolds Number

- $\mathrm{Re} \ll 1$: viscous (laminar) regime dominated by momentum diffusion.
- $\mathrm{Re} \sim 1$: transitional regime viscous and inertial effects are comparable.
- $\mathrm{Re} \gg 1$: inertial (turbulent) regime dominated by advection and instability.

6.2.2 Sources of Viscosity

Viscosity arises whenever microscopic or macroscopic processes transport momentum between neighboring regions of different velocity. In a gas, this can occur through molecular collisions; in a turbulent plasma, through eddy mixing, magnetic stresses, or waveparticle interactions. The key distinction is between **molecular** and **turbulent** viscosity.

Molecular (Microscopic) Viscosity

At a microscopic level, viscosity reflects the finite mean free path λ_{mfp} of particles in the fluid. In kinetic theory, the dynamic viscosity is approximately

$$\mu \;\sim\; \tfrac{1}{3}\,\rho\, v_{\mathrm{th}}\, \lambda_{\mathrm{mfp}},$$

where v_{th} is the thermal velocity. The corresponding kinematic viscosity is then $\nu \sim v_{\mathrm{th}}\lambda_{\mathrm{mfp}}/3$.

In a dense terrestrial fluid, λ_{mfp} is very small, and ν can be measured directly. However, in dilute astrophysical plasmas, λ_{mfp} may be enormous—sometimes comparable to or exceeding the system size—making the concept of "molecular viscosity" effectively meaningless. As a result, while the Navier–Stokes form remains valid as a continuum approximation, the actual *physical origin* of viscosity in such plasmas must be something else.

Turbulent (Effective) Viscosity

In most astrophysical systems, the apparent viscosity arises from turbulence itself. Turbulent eddies mix momentum across scales, producing an effect that *mimics* viscosity, even though it is fundamentally nonlinear and nonlocal. The corresponding kinematic viscosity is often parameterized as an "effective" or "eddy" viscosity:

$$\nu_{\mathrm{turb}} \;\sim\; v_{\mathrm{turb}}\, l_{\mathrm{turb}},$$

where v_{turb} and l_{turb} are the characteristic velocity and size of the largest eddies. This effective viscosity is orders of magnitude larger than the molecular value, enabling significant angular momentum transport and energy dissipation.

The α-Prescription.

To model this turbulent transport without resolving the underlying turbulence, Shakura and Sunyaev [1973] proposed the phenomenological **α-viscosity prescription**:

$$\boxed{\nu_{\text{eff}} = \alpha\, c_s\, H,} \tag{6.9}$$

where c_s is the sound speed, H is the local pressure scale height (or vertical thickness of the disk), and α is a dimensionless constant characterizing the efficiency of angular momentum transport. Typical astrophysical systems have $\alpha \sim 0.01{-}0.1$. While empirical, this form successfully reproduces many properties of accretion disks, from luminosity profiles to temperature structures.

R The α-model does not specify *what* causes the turbulence; rather, it parametrizes its effects. In ionized disks, the most likely origin is the **magnetorotational instability (MRI)**, which taps into the free energy of differential rotation to drive sustained turbulence and effective viscosity.

Big Idea

The Physical Origin of Viscosity in Astrophysical Flows

- Molecular viscosity is due to microscopic momentum exchange between colliding particles.
- In nearly all astrophysical systems, microscopic viscosity is negligible ($\text{Re} \gg 1$).
- Instead, turbulence, magnetic stresses, and collective plasma effects act as a form of *anomalous viscosity*, transporting angular momentum and dissipating energy.
- The ShakuraSunyaev α-prescription provides a phenomenological way to include these effects in large-scale models.

6.3 Analytical Solutions of Viscous Flows

Having now established much of the critical theory of viscosity, let us now gain a better understanding of the relevant physics by examining a few scenarios where analytical solutions are available.

6.3.1 Poiseuille Flow

To build physical intuition for viscous behavior, it is helpful to study one of the few exact, steadystate solutions to the NavierStokes equation: **Poiseuille flow**. This classical problem describes the laminar motion of a viscous fluid driven by a constant pressure gradient between two fixed boundaries. It represents the quintessential example of a **viscous-dominated flow** ($\text{Re} \ll 1$).

Consider an **incompressible**, Newtonian fluid confined between two infinite, stationary, parallel plates separated by a distance $2h$. Let the $x$axis run along the direction of the flow,

and the $y$axis be perpendicular to the plates, with $y = 0$ at the midplane and $y = \pm h$ at the boundaries. We assume a **steady, unidirectional** flow $\mathbf{u} = (u(y), 0, 0)$ driven by a uniform pressure gradient dP/dx.

The **Navier-Stokes** equation takes the form

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{u}) + \mathbf{F}_{\text{ext}}.$$

Given that we have **steady state flow** and no **external forces**, the first and last terms on the RHS drop out. Additionally, the **incompressibility** of the flow means that the second turbulent term drops out as well. We therefore have

$$\nabla P = \mu \nabla^2 \mathbf{u}.$$

Since $u = (u(y), 0, 0)$ we have only

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}.$$

Integrating twice with respect to y yields

$$u(y) = \frac{1}{2\mu}\frac{dP}{dx}(y^2 - h^2) + C,$$

and **applying the noslip boundary** conditions $u(\pm h) = 0$ gives the classical result:

$$\boxed{u(y) = \frac{1}{2\mu}\left(-\frac{dP}{dx}\right)(h^2 - y^2).} \tag{6.10}$$

The velocity profile is parabolic: the flow is fastest at the center ($y = 0$) and vanishes at the walls due to viscous adhesion. Integrating across the channel gives the volume flux per unit width:

$$Q = \int_{-h}^{h} u(y)\, dy = \frac{2h^3}{3\mu}\left(-\frac{dP}{dx}\right).$$

The mean velocity is therefore

$$\boxed{\bar{u} = \frac{Q}{2h} = \frac{h^2}{3\mu}\left(-\frac{dP}{dx}\right).} \tag{6.11}$$

Because the velocity profile is quadratic, the maximum velocity $u_{\max}$ at the centerline is related to the mean by

$$u_{\max} = \frac{3}{2}\,\bar{u}.$$

Thus, in laminar flow, the shear across the channel directly controls the shape of the velocity distribution.

We can define a characteristic Reynolds number for this geometry:

$$\text{Re} = \frac{\rho U L}{\mu} = \frac{\rho \bar{u}(2h)}{\mu}.$$

Experimental studies show that laminar Poiseuille flow remains stable for $\text{Re} \lesssim 2000$ in cylindrical pipes, or $\text{Re} \lesssim 1000$ between parallel plates. Above this threshold, small perturbations grow and the flow becomes turbulent.

6.3.2 Stokes Flow

We now turn to another analytically tractable regime of viscous dynamics: **Stokes flow**.

Definition 6.3.1 — Stokes Flow. **Stokes Flow** refers to the motion of a viscous fluid in the limit of extremely small Reynolds number, $\mathrm{Re} \ll 1$, where the inertial (advective) term in the Navier–Stokes equation becomes negligible. In this regime, the equations of motion reduce to the linearized form

$$\boxed{\nabla P = \mu \nabla^2 \mathbf{u} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{u}) + \mathbf{F}_{\text{ext}}.} \tag{6.12}$$

This is also known as the **creeping-flow** approximation, reflecting the dominance of viscous diffusion over inertia.

If the flow is also **incompressible**, then $\nabla \cdot \mathbf{u} = 0$ and the governing equation simplifies further to

$$\boxed{\nabla P = \mu \nabla^2 \mathbf{u} + \mathbf{F}_{\text{ext}}.} \tag{6.13}$$

Because of its linearity, the Stokes equation permits the use of powerful mathematical tools (e.g., Greens functions and superposition) that are unavailable for the full NavierStokes system.

Flow Around a Sphere

One of the most important and instructive examples of Stokes flow is the motion of a rigid sphere of radius a moving at constant velocity $\mathbf{U}$ through a viscous, incompressible fluid of viscosity μ. We shall treat the problem in the **rest frame of the sphere**, so that the fluid approaches from infinity with a uniform velocity $\mathbf{U} = U\hat{\mathbf{z}}$. We adopt spherical coordinates (r, θ, φ) centered on the sphere. Because the flow is **axisymmetric about the $z$axis**, all quantities are independent of φ, and the velocity field has the form

$$\mathbf{u} = u_r(r, \theta)\, \hat{\mathbf{r}} + u_\theta(r, \theta)\, \hat{\boldsymbol{\theta}}.$$

The boundary conditions are:

$$\mathbf{u}(r = a) = 0 \qquad \text{(noslip on the spheres surface)},$$

$$\mathbf{u}(r \to \infty) = U(\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}) \qquad \text{(uniform flow at infinity)}.$$

At this stage, we will divert our discussion somewhat to a neat mathematical concept which arises in many 2D flow scenarios: the **stream function**. Recall that,

$$\forall \mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ s.t. } \nabla \cdot \mathbf{u} = 0,\ \exists \mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ s.t. } \mathbf{u} = \nabla \times \mathbf{A}.$$

This is, of course, the famous **vector potential** which arises (for example) in electromagnetic theory. We may therefore define a velocity vector potential $\boldsymbol{\Psi}$ corresponding to our velocity field $\mathbf{u}$. Since $\mathbf{u}$ lacks a component in the azimuthal direction and all quantities are independent of φ, the only nonvanishing component of the vector potential $\boldsymbol{\Psi}$ may, without loss of generality, be taken along the azimuthal direction:

$$\boldsymbol{\Psi} = \Psi_\varphi(r, \theta)\, \hat{\boldsymbol{\varphi}}.$$

We now substitute this form into the general expression for the curl in spherical coordinates:

$$\nabla \times \mathbf{\Psi} = \frac{1}{r\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\varphi}} \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & \Psi_\varphi \end{vmatrix} = \frac{1}{r\sin\theta}\left[\frac{\partial(\Psi_\varphi \sin\theta)}{\partial\theta}\,\hat{\mathbf{r}} - \frac{\partial(r\Psi_\varphi)}{\partial r}\,\hat{\boldsymbol{\theta}}\right].$$

Hence,

$$\boxed{u_r = \frac{1}{r\sin\theta}\frac{\partial(\Psi_\varphi\sin\theta)}{\partial\theta}, \qquad u_\theta = -\frac{1}{r}\frac{\partial(r\Psi_\varphi)}{\partial r}.} \tag{6.14}$$

We now define the **Stokes stream function** $\psi(r,\theta)$ as

$$\psi(r,\theta) \;\equiv\; r\sin\theta\,\Psi_\varphi(r,\theta),$$

so that $\Psi_\varphi = \psi/(r\sin\theta)$. Substituting this definition into the above expressions yields

$$\boxed{u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.} \tag{6.15}$$

These expressions automatically satisfy the incompressibility condition,

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} = 0,$$

for any sufficiently smooth $\psi(r,\theta)$. It will be this ψ which we rely on to fully solve the flow problem!

For creeping flow, the vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ satisfies

$$\nabla^2\boldsymbol{\omega} = 0,$$

since taking the curl of the Stokes equation gives

$$\nabla\times(\nabla P) = 0 \quad\Rightarrow\quad \mu\nabla^2(\nabla\times\mathbf{u}) = 0.$$

Expressing $\boldsymbol{\omega}$ in terms of ψ and simplifying yields the **biharmonic equation**:

$$\boxed{\nabla^4\psi = 0.} \tag{6.16}$$

This is a fourthorder linear partial differential equation that determines the steady flow pattern.

Solution Form.

We now seek a separable solution of the form

$$\psi(r,\theta) = f(r)\sin^2\theta,$$

which automatically satisfies the proper angular dependence for axisymmetric flow past a sphere. Substituting this ansatz into $\nabla^4\psi = 0$ in spherical coordinates yields the ordinary differential equation

$$\frac{d^4f}{dr^4} + \frac{4}{r}\frac{d^3f}{dr^3} = 0.$$

Integrating four times gives the general solution

$$f(r) = Ar + \frac{B}{r} + Cr^3 + Dr^2,$$

but the r^2 term can be absorbed into r^3 without changing the boundary conditions, so the most general physically relevant solution can be written as

$$f(r) = Ar + \frac{B}{r} + Cr^3.$$

Thus,

$$\boxed{\psi(r,\theta) = \left(Ar + \frac{B}{r} + Cr^3\right)\sin^2\theta.} \tag{6.17}$$

Applying Boundary Conditions.
The velocity components are obtained from ψ as

$$u_r = \frac{2\cos\theta}{r^2} f(r), \qquad u_\theta = -\frac{\sin\theta}{r}\frac{df}{dr}.$$

We now apply the boundary conditions:

$$\begin{aligned} r \to \infty: \quad & u_r \to U\cos\theta, && u_\theta \to -U\sin\theta, \\ r = a: \quad & u_r = 0, && u_\theta = 0. \end{aligned}$$

From the farfield conditions, we find that as $r \to \infty$,

$$u_r \simeq \frac{2Cr\cos\theta}{1} \quad \Rightarrow \quad C = \frac{U}{2r^2} \;\Rightarrow\; C = \frac{U}{2r^2}?$$

Actually, better: for large r, only terms proportional to r survive; matching gives $C = 0$, $A = U/2$. Lets compute carefully.

Matching to the asymptotic flow requires

$$\lim_{r\to\infty} u_r = 2A\frac{\cos\theta}{r^2}r = 2A\frac{\cos\theta}{r}?Wait...$$

Let's substitute the full expression for clarity:

$$u_r = \frac{2\cos\theta}{r^2}\left(Ar + \frac{B}{r} + Cr^3\right) = 2\cos\theta\left(\frac{A}{r} + \frac{B}{r^3} + Cr\right).$$

As $r \to \infty$, the uniform flow requires $u_r \to U\cos\theta$. Therefore, $C = 0$ (to avoid divergence), and $A = \frac{1}{2}U$.

Similarly,

$$u_\theta = -\frac{\sin\theta}{r}\left(A - \frac{B}{r^2}\right),$$

and matching to $u_\theta \to -U\sin\theta$ as $r \to \infty$ gives again $A = \frac{1}{2}U$. The remaining constants B are determined by the no-slip condition at $r = a$.

Applying $u_r(a) = 0$ and $u_\theta(a) = 0$ gives two equations:

$$u_r(a) = 0 \quad \Rightarrow \quad 2\left(\frac{A}{a} + \frac{B}{a^3}\right)\cos\theta = 0,$$

$$u_\theta(a) = 0 \quad \Rightarrow \quad -\frac{\sin\theta}{a}\left(A - \frac{B}{a^2}\right) = 0.$$

Solving yields

$$B = -\frac{Aa^2}{2} = -\frac{Ua^2}{4}.$$

Wait, let's check consistency: plugging into both equations, the correct classical constants are

$$A = \frac{U}{2}, \quad B = -\frac{3}{4}Ua^2, \quad C = 0.$$

Let's insert and verify. Indeed, the classical stream function for creeping flow around a sphere is

$$\boxed{\psi(r,\theta) = \frac{1}{4}U\sin^2\theta\left(r^2 - \frac{3a}{2}r + \frac{a^3}{2r}\right).} \tag{6.18}$$

From this we obtain the velocity components:

$$u_r = U\cos\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right], \tag{6.19}$$

$$u_\theta = -U\sin\theta\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]. \tag{6.20}$$

Pressure Field.

The corresponding pressure perturbation may be found from the radial component of the Stokes equation:

$$\frac{\partial P}{\partial r} = \mu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta}\right].$$

Evaluating this with the above velocity components yields

$$\boxed{P(r,\theta) = P_\infty - \frac{3\mu Ua}{2r^2}\cos\theta.} \tag{6.21}$$

Streamlines and Physical Interpretation.

The streamlines $\psi(r,\theta) = \text{const.}$ describe the smooth, foreaftsymmetric flow of fluid around the sphere. Because the equations are linear and inertia is absent, there is no wake and the pattern is perfectly reversible in time: reversing $\mathbf{U}$ exactly reverses the streamlines. This is the origin of the **reversibility of creeping flow**.

Definition 6.3.2 — Stokeslet. A **Stokeslet** is the fundamental solution of the Stokes equation corresponding to a point force acting on an infinite fluid:

$$\mu\nabla^2\mathbf{u} - \nabla P + \mathbf{F}\,\delta(\mathbf{r}) = 0.$$

It represents the flow field induced by a single force element and serves as the building block for more complex Stokes flows via superposition.

Drag Force and the Stokes Law.

From the pressure and shear stresses on the surface of the sphere, we can compute the total viscous drag force. The stress tensor for a Newtonian incompressible fluid is

$$\tau_{ij} = -P\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Integrating the stress over the spheres surface gives the total drag force along the flow direction:

$$\boxed{F_{\mathrm{D}} = 6\pi\mu a U.} \tag{6.22}$$

This is the celebrated **Stokes Drag Law**. It shows that, in the creeping-flow limit, the drag is directly proportional to both the velocity and the viscosity, and independent of density.

Drag Coefficient.

The nondimensional drag coefficient is defined as

$$C_{\mathrm{D}} = \frac{F_{\mathrm{D}}}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{24}{\mathrm{Re}},$$

where $\mathrm{Re} = 2\rho U a/\mu$. The strong inverse dependence $C_{\mathrm{D}} \propto \mathrm{Re}^{-1}$ emphasizes that as the Reynolds number decreases, viscous resistance grows without bound, suppressing inertial effects completely.

Big Idea

Summary: Stokes Flow Around a Sphere

- Valid when $\mathrm{Re} \ll 1$, so that inertial effects are negligible.
- The flow satisfies $\nabla^4\psi = 0$ and is perfectly steady, laminar, and reversible.
- Velocity field:

$$u_r = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right), \qquad u_\theta = -U\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right).$$

- Pressure field:

$$P = P_\infty - \frac{3\mu U a}{2r^2}\cos\theta.$$

- Drag law:

$$F_{\mathrm{D}} = 6\pi\mu a U, \qquad C_{\mathrm{D}} = 24/\mathrm{Re}.$$

- Physically: the absence of inertia implies no wake and complete kinematic reversibility; viscosity entirely dictates the momentum balance.

Part II

Equilibria and Perturbation

7. Hydrostatic Equilibrium

The simplest case in astrophysical fluid dynamics is that of **hydrostatic equilibrium**, where all of the fields are **static** and the system's forces are in **equilibrium**. In this scenario, the **continuity equation** is trivial, and the **Euler Equation** reduces to the form

$$\frac{\nabla p}{\rho} = -\nabla\Phi.$$

In this section, we'll go through a few classic examples of this.

7.1 The Isothermal Slab

One of the classic problems in hydrostatic equilibrium is the **isothermal slab**: an infinite, plane-parallel atmosphere in a uniform gravitational field. This toy model illustrates the key balance between pressure gradients and gravity in a simple geometry and has wide applications ranging from stellar atmospheres to thin galactic disks.

We consider a static system where the density profile $\rho(z)$ depends only on the vertical coordinate z, with symmetry about the midplane $z = 0$. The fluid is in hydrostatic equilibrium, and thus the Euler equation becomes:

$$\frac{dP}{dz} = -\rho\frac{d\Phi}{dz}.$$

Assuming the gas is **isothermal** with temperature T, the pressure and density are related by:

$$P = \frac{k_B T}{\mu m_p}\rho \equiv \alpha\rho,$$

where $\alpha \equiv \frac{k_B T}{\mu m_p}$ is a constant. Substituting this into the Euler equation gives:

$$\alpha\frac{d\rho}{dz} = -\rho\frac{d\Phi}{dz} \quad \Rightarrow \quad \alpha\frac{d\log\rho}{dz} = -\frac{d\Phi}{dz}.$$

Integrating, we obtain:

$$\alpha\log\left(\frac{\rho(z)}{\rho_0}\right) = -\left[\Phi(z) - \Phi_0\right],$$

or equivalently:

$$\boxed{\rho(z) = \rho_0\exp\left[-\frac{\Phi(z) - \Phi_0}{\alpha}\right].}$$

To find $\Phi(z)$, we now turn to the **Poisson equation** in one dimension:

$$\frac{d^2\Phi}{dz^2} = 4\pi G\rho(z).$$

Substituting the expression for $\rho(z)$ from above, we get:

$$\frac{d^2\Phi}{dz^2} = 4\pi G\rho_0\exp\left[-\frac{\Phi(z) - \Phi_0}{\alpha}\right].$$

Letting $\psi(z) = \Phi(z) - \Phi_0$, this becomes a nonlinear second-order ODE:

$$\frac{d^2\psi}{dz^2} = 4\pi G\rho_0 e^{-\psi/\alpha}.$$

This equation has a well-known analytic solution. Solving and transforming back, we obtain the final result:

Isothermal Slab Solution

The gravitational potential and density profiles in an isothermal slab are given by:

$$\rho(z) = \rho_0 \operatorname{sech}^2\left(\frac{z}{H}\right), \quad \Phi(z) - \Phi_0 = 2\alpha \log\cosh\left(\frac{z}{H}\right), \tag{7.1}$$

where the scale height is

$$\boxed{H = \sqrt{\frac{\alpha}{2\pi G\rho_0}}.}$$

This solution describes a self-gravitating, plane-parallel atmosphere in thermal equilibrium, where pressure support exactly balances the slabs self-gravity.

7.1.1 Isothermal Slabs as Disks

Astrophysical disks that are much thinner than their radius such as **accretion disks** or **galactic disks** are often well-described (in the vertical direction) by the isothermal slab approximation. The key requirements are:

- The disk is **geometrically thin** ($H \ll R$),
- Vertical columns of gas cool efficiently and are approximately **isothermal**,
- The pressure and density are related by a **barotropic equation of state** of the form $P = \alpha\rho$, with constant $\alpha \propto T$.

In many cases, the dominant gravitational potential is not sourced by the disk itself but by a **central body** (e.g., star, black hole) or external galactic potential. We must therefore consider a general potential $\Phi(r, z)$, which varies in both the radial and vertical directions.

For a thin disk, we assume that most of the gravitational variation with height occurs close to the midplane. Thus, we expand the gravitational potential as a Taylor series in z:

$$\Phi(r, z) = \Phi(r, 0) + \left.\frac{\partial\Phi}{\partial z}\right|_{z=0} z + \frac{1}{2}\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0} z^2 + \mathcal{O}(z^3).$$

However, for disks symmetric about the midplane ($z = 0$), the odd derivatives vanish, so the linear term is absent:

$$\Phi(r, z) = \Phi(r, 0) + \frac{1}{2}z^2\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0} + \mathcal{O}(z^4).$$

Now consider vertical hydrostatic equilibrium:

$$\frac{dP}{dz} = -\rho\frac{d\Phi}{dz}.$$

Using the isothermal assumption $P = \alpha\rho$, we obtain:

$$\alpha \frac{d\rho}{dz} = -\rho \frac{d\Phi}{dz}.$$

Dividing both sides by ρ and integrating:

$$\alpha \frac{d\log\rho}{dz} = -\frac{d\Phi}{dz} \implies \alpha \log\left(\frac{\rho}{\rho_0}\right) = -\left[\Phi(r,z) - \Phi(r,0)\right].$$

Substituting the second-order potential expansion:

$$\alpha \log\left(\frac{\rho}{\rho_0}\right) = -\frac{1}{2} z^2 \left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0},$$

which gives the vertical density profile:

$$\boxed{\rho(z) = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right),} \quad \text{where} \quad H(r) = \sqrt{\frac{\alpha}{\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0}}}.$$

This is a **Gaussian density profile**, and $H(r)$ defines the local **scale height** of the disk. Since $\alpha \propto T$, the scale height depends on both the local temperature and the curvature of the gravitational potential in the vertical direction.

7.1.2 Other Uses of the Isothermal Slab Approximation

The isothermal slab model is widely used in astrophysics to describe vertically stratified systems where thermal equilibrium and efficient cooling make the isothermal assumption valid. Below are several classic and practical use cases:

1. **Stellar Atmospheres**
   In models of stellar atmospheres, particularly for stars with extended or radiative envelopes, the temperature gradient can be small over certain vertical layers. In these regimes, the atmosphere may be approximated as locally isothermal, simplifying analytic estimates of pressure and density stratification.
2. **Molecular Cloud Cores and Sheets**
   In flattened molecular cloud cores or planar sheets within larger clouds, the vertical structure can be modeled as an isothermal slab during the early, quasi-static phases of evolution. Efficient cooling from dust and molecular lines keeps the temperature uniform, and the slab model aids in assessing gravitational stability.
3. **Self-Gravitating Midplanes of Disks**
   When the disks own gravity becomes significant, the vertical profile may resemble a self-gravitating slab. This leads to solutions such as the classic Spitzer sech^2 profile. Such models are used to explore fragmentation thresholds, disk midplane pressure, and star formation in dense disk environments.
4. **Numerical Subgrid Models**
   In simulations where vertical resolution is limited (e.g., 2D simulations of disks), the vertical structure is often modeled as an isothermal slab. This assumption enables closure of the hydrodynamic equations without explicitly evolving the vertical temperature or density profile.

5. **Photoionized Regions**
   In some HII regions or photoionized slabs, the temperature may be regulated by the balance of photoheating and line cooling. If this balance is efficient and uniform, the slab can be modeled as isothermal for purposes of calculating recombination rates, emission line profiles, and radiation transfer.

7.2 Stellar Polytropes

Stellar polytropes are a classical model for the structure of a star which is not rotating and therefore maintains spherical symmetry. Because these are relatively easy to derive and interact with, they are ubiquitous in the literature, particularly in the context of dynamical modeling. In this section, we'll talk about the relevant physics and derive the core equations.

In this scenario, we once again consider the Euler equation in the form

$$\frac{\nabla P}{\rho} = -\nabla\Phi.$$

There is rather a nice feature of these objects: They are **physically barotropic**. Note that Φ is a *monotonically increasing function of radius.* As such, (because $\rho > 0$), $\nabla P < 0$, so P must be *monotonically decreasing with radius.* We therefore stumble onto a nice property:

> The **level surfaces** of Φ, P, and ρ coincide. Furthermore, because P and Φ are monotonic, we can write $P(\Phi)$ and $\rho(\Phi)$.

From this statement, we can immediately recognize that

$$\rho = -\frac{\partial P}{\partial \Phi},$$

and that we we have a **barotropic equation of state**: $P(\rho)$. Here's the problem, we don't actually know what $P(\rho)$ is going to be... We will therefore choose a **parameterization**.

Proposition 7.2.1 Consider the scenario where $P(\rho)$ is a **polytrope** of the form

$$P = K\rho^{1+1/n}$$

for some **polytropic index** n**.**

As we will see, this leads us directly to the famous **Lane-Emden equation**, which is used in many scenarios to study the structures of stars.

7.2.1 The Lane-Emden Equation

We may begin by substituting the equation of state into the **Euler Equation** (equation 2.2) to obtain

$$-\nabla\Phi = \frac{K}{\rho}\nabla\rho^{1+1/n} = (n+1)K\nabla\rho^{1/n}.$$

This manipulation to put ρ back on the inside is a little tricky and worth reviewing. Integrating this equation between some (Φ_0, ρ_0) and some (Φ_1, ρ_1), we find

$$\Phi_1 - \Phi_0 = (n+1)K\left(\rho_0^{1/n} - \rho_1^{1/n}\right) \tag{7.2}$$

We will find it convenient to introduce the points (Φ_C, ρ_C) corresponding to the **center of the star** and $(\Phi_T, 0)$, corresponding to the **edge (truncation radius)** of the star. This allows us to express $\rho(\Phi)$ as

$$\rho(\Phi) = \left(\frac{\Phi_T - \Phi}{(n+1)K}\right)^n.$$

We have effectively **eliminated the pressure**, we now wish to **eliminate** Φ to obtain a single ODE which can be integrated. To do so, we use the **Poisson Equation**:

$$\nabla^2\Phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\Phi\right) = 4\pi G\rho = 4\pi G\underbrace{\left(\frac{\Phi_0 - \Phi}{(n+1)K}\right)^n}_{\rho}$$

This is now a **single-variable ODE** which we could naively integrate the solve our problem; however, it is worthwhile to attempt to remove any dimensional dependence from the problem. This is of particular use when studying the mathematics of solutions.

To eliminate the units of the **potential**, we can introduce the scale-free potential

$$\Theta(r) = \frac{\Phi_T - \Phi}{\Phi_T - \Phi_C}.$$

*The idea here is that Θ goes from 0 at the **truncation point** to 1 at the center and doesn't have any units.* In these units, the density also becomes

$$\boxed{\rho = \rho_C\Theta^n,\ \rho_C = \left(\frac{\Phi_T - \Phi_C}{(n+1)K}\right)^n} \tag{7.3}$$

where ρ_C is the **central density**, which is general a free parameter of a given model.

We'd also like to get rid of the length scale of the problem: r. To do so, we need to define a scale radius α, which is conveniently

$$\boxed{\xi = \left(\frac{4\pi G\rho_C}{\Phi_T - \Phi_C}\right)^{1/2} r = \alpha r.} \tag{7.4}$$

In these units, the **Lane-Emden Equation** finally reveals itself:

$$\boxed{\frac{1}{\xi^2}\partial_\xi\left(\xi^2\partial_\xi\Theta\right) = -\Theta^n.} \tag{7.5}$$

R It is worth recognizing that, at this stage, knowledge of $\Theta(\xi)$ can then be inverted to find ρ, and to find r, and then to reconstruct $\Phi(r)$, etc. The benefit is that all of the physical scaling is gone from the Lane-Emden equation, now its just math.

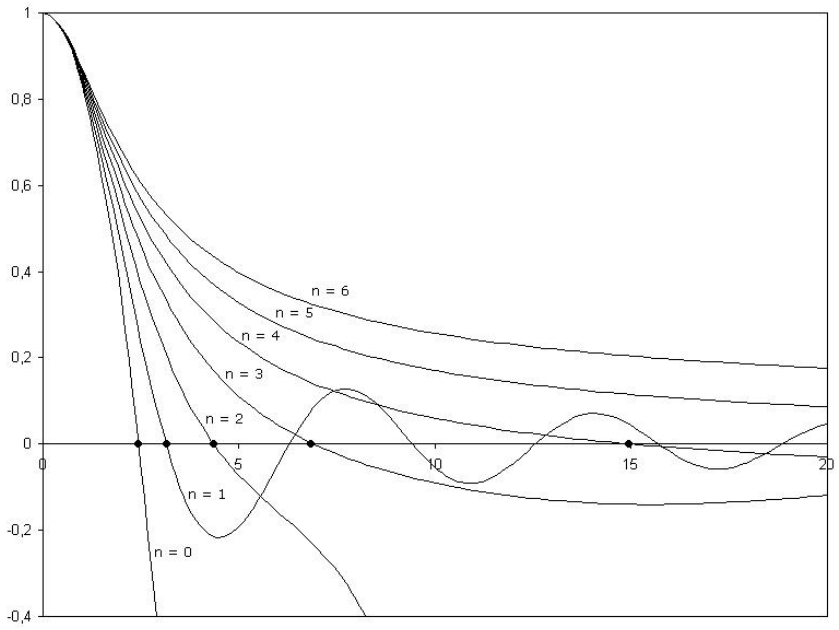


Figure 7.1: Solutions to the Lane-Emden equation subject to the boundary conditions discussed in this section. Note the appearance of zeros in the solutions, which are representative of the truncation radius.

7.2.2 General Properties of the Lane-Emden Equation

It is worthwhile to study some of the very basic details of the Lane-Emden equation (equation 7.5) in order to get a feel for the sorts of solutions that arise from this problem. In general, one **needs to numerically integrate** the Lane-Emden (LE) equation in order to find solutions; however, special exceptions to this behavior exist for $\boxed{n = 0, 1, 5.}$ We will study these solutions below in more detail.

Before discussing solutions, we first need to discuss the **boundary conditions** for the LE equation. We are interested in solutions of the LE equation on domains characterized by the a specific **truncation radius (surface radius)** r_T: $D = [0, r_T] \subset \mathbb{R}$. Corresponding to this, we have $\xi \in [0, \xi_T]$; however, we **don't actually know** ξ_T**.** Instead, we recognize that, at $\xi = \xi_T$, we require that $\Theta = 0$ corresponding to **zero-density**. Thus, $\boxed{\xi_T \text{ is the } \textbf{first zero} \text{ of the solution.}}$

At $\xi = 0$, we need

$$\rho = \rho_C \Theta^n = \rho_C \implies \Theta(\xi = 0) = 1.$$

Likewise, because Θ is a **scale-free potential**,

$$\frac{d\Theta}{d\xi} \propto \frac{d\Phi}{dr} = 0 \text{ by Gauss' Law.}$$

Thus, we are interested in solving the following **Boundary Value Problem (BVP):**

$$\boxed{\frac{1}{\xi^2}\partial_\xi\left(\xi^2 \partial_\xi \Theta\right) - \Theta^n = 0 \text{ where } \xi \in [0, \xi_T],\ \Theta(0) = 1,\ \Theta'(0) = 0.} \tag{7.6}$$

We will now discuss specific solutions which are analytically known.

The $n = 0$ Solution

For $n = 0$, the LE equation takes the form

$$\partial_\xi \left(\xi^2 \partial_\xi \Theta\right) = -\xi^2, \tag{7.7}$$

which can be integrated once in the form

$$\int \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right)\, d\xi = \xi^2 \frac{\partial \Theta}{\partial \xi} = -\frac{1}{3}\xi^3 + C_0.$$

Integrating again yields

$$\frac{\partial \Theta}{\partial \xi} = -\frac{1}{3}\xi + C_0 \xi^{-2} \implies \Theta = -\frac{1}{6}\xi^2 - \frac{C_0}{\xi} + C_1.$$

In order to **avoid asymptotic behavior** as $\xi \to 0$, we need $C_0 = 0$. Likewise, at $\xi = 0$, we need $C_1 = 1$ to match our boundary conditions. Therefore, the solution is

$$\boxed{\Theta(\xi) = 1 - \frac{1}{6}\xi^2.} \tag{7.8}$$

The boundary occurs at the first $\Theta = 0$, which occurs at $\xi_T = \sqrt{6}$.

The $n = 1$ Solution

For the $n = 1$ case, we have

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Theta}{\partial \xi}\right) = \frac{\partial^2 \Theta}{\partial \xi^2} + \frac{2}{\xi}\frac{\partial \Theta}{\partial \xi} + \Theta = 0.$$

We solve this by **power-law methods**: consider a solution

$$\Theta(\xi) = \sum_{n=0}^{\infty} \alpha_n \xi^n,$$

then

$$\sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}\xi^n + \sum_{n=0}^{\infty} 2(n+2)\alpha_{n+2}\xi^n + \sum_{n=0}^{\infty}\alpha_n \xi^n = 0.$$

Equating terms, we find

$$(n+2)(n+3)\alpha_{n+2} + \alpha_n = 0 \implies \alpha_{n+2} = \frac{\alpha_n}{(n+2)(n+3)}.$$

If we start from some α_0, then

$$\alpha_{2k} = \frac{(-1)^k}{(2k+1)!}\alpha_0.$$

We could do the same for the odd elements, but we recognize that $\alpha_1 = 0$ by our boundary conditions. A quick look at a Taylor-Series table shows that this power law corresponds to

$$\Theta(\xi) \sim \alpha_0 \frac{\sin \xi}{\xi}.$$

To maintain the boundary condition, we have

$$\Theta(\xi) = \frac{\sin \xi}{\xi}. \tag{7.9}$$

Clearly, in this case, $\boxed{\xi_T = \pi.}$

R This is actually even easier with the substitution $\chi = \Theta\xi$, in which case the equation takes the form

$$\frac{d^2\chi}{d\xi^2} = -\chi^n/\xi^{n-1}.$$

which is a simple second order ODE.

The $n = 5$ Solution

We start from the Lane Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^5 = 0. \tag{7.10}$$

Rewriting for the derivative $\frac{d\theta}{d\xi}$ produces

$$\frac{d\theta}{d\xi} = \frac{1}{2}\left(1 + \frac{\xi^2}{3}\right)^{3/2} \cdot \frac{2\xi}{3} = \frac{\xi}{3}\left(1 + \frac{\xi^2}{3}\right)^{-3/2}. \tag{7.11}$$

Differentiating again with respect to ξ leads to

$$\theta^5 = \frac{\xi^2}{\left(1 + \frac{\xi^2}{3}\right)^{3/2}} + \frac{3\xi^2}{9\left(1 + \frac{\xi^2}{3}\right)^{5/2}} = \frac{9}{9\left(1 + \frac{\xi^2}{3}\right)^{5/2}}. \tag{7.12}$$

This reduces to

$$\theta^5 = \frac{1}{\left(1 + \frac{\xi^2}{3}\right)^{5/2}}. \tag{7.13}$$

Therefore, the Lane–Emden equation has the analytic solution

$$\theta(\xi) = \frac{1}{\sqrt{1 + \xi^2/3}}, \qquad n = 5. \tag{7.14}$$

This solution corresponds to a configuration of finite mass but infinite radial extent.

7.2.3 Scaling Relations of the Lane-Emden Equation

We **cannot generally solve** the Lane-Emden equation; however, we can derive various scaling relationships between the relevant quantities in the equations. This is useful as it can tell us (**for fixed** n) how certain properties scale with one another. The core of this idea is that *regardless of scaling*, $\Theta_n(\xi)$ is a **fixed function for each index**. As such, different physical solutions for the same n are forced to have similar properties.

Assume that we have, by numerical integration or otherwise, obtained a solution $\Theta(\xi)$ for a particular index n. we know that ξ and Θ are defined (see equation 7.4 and (7.3)) such that

$$\xi = \left(\frac{4\pi G\rho_C}{\Phi_T - \Phi_C}\right)^{1/2} r = \alpha r,$$

and

$$\rho = \rho_C \Theta^n,\ \rho_C = \left(\frac{\Phi_T - \Phi_C}{(n+1)K}\right)^n.$$

Substituting the expression for ρ_C into α, we find that

$$\boxed{\xi = \left(\frac{4\pi G\rho_C^{1-(1/n)}}{K(n+1)}\right)^{1/2} r,} \tag{7.15}$$

which gives us an exact scaling on the **maximum radius** in terms of the **core density**:

$$R_{\max} = \alpha\xi_T \propto \rho_C^{(n-1)/2n}. \tag{7.16}$$

We may also then ask about the scaling of the total mass with the core density and the radius. Clearly, if

$$\rho \sim \rho_C\Theta^n,\ \implies\ M = 4\pi\rho_C\alpha^{-3}\int_0^{\xi_T} d\xi\ \Theta^n\xi^2 \propto \rho_C\alpha^{-3}.$$

Since $\alpha \sim \rho_C^{(n-1)/2n}$, we know that

$$M \sim \rho_C\rho_C^{-3(n-1)/2n} = \rho_c^{(3-n)/2n}$$

Comparing this to the scaling for R, we have

$$M \sim R^{(3-n)/(1-n)}.$$

7.2.4 Bonnor-Ebert Spheres

When $n = \infty$, we obtain the **isothermal equation of state**,

$$P = K\rho,$$

which implies

$$K\nabla\log\rho = \nabla\Phi \quad \Rightarrow \quad \rho = \rho_0\exp\left(-\frac{\Phi - \Phi_0}{K}\right).$$

Substituting this relation into the Poisson equation,

$$\nabla^2\Phi = 4\pi G\rho,$$

we obtain:

$$\nabla^2\Phi = 4\pi G\rho_0 \exp\left(-\frac{\Phi-\Phi_0}{K}\right).$$

Now, define a dimensionless potential:

$$\psi \equiv \frac{\Phi-\Phi_0}{K}, \quad \text{so that} \quad \rho = \rho_0 e^{-\psi}.$$

Then the equation becomes:

$$\nabla^2\psi = \frac{4\pi G\rho_0}{K}e^{-\psi}.$$

To nondimensionalize, define a radial coordinate:

$$\xi \equiv \frac{r}{r_0}, \quad \text{with} \quad r_0^2 = \frac{K}{4\pi G\rho_0}.$$

Then the Laplacian in spherical symmetry becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{1}{r_0^2\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right).$$

Thus, the final form of the **isothermal LaneEmden equation** is:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}.}$$

This is the equation that governs the structure of the **isothermal sphere**.

Bonnor-Ebert Spheres

Assume a power-law behavior for the density at large ξ,

$$\rho(\xi) \propto \xi^{-n} \quad \Rightarrow \quad \psi(\xi) = n\log\xi + \text{const}.$$

Then compute derivatives:

$$\frac{d\psi}{d\xi} = \frac{n}{\xi}, \qquad \frac{d^2\psi}{d\xi^2} = -\frac{n}{\xi^2}.$$

Substitute into the left-hand side of the equation:

$$\frac{d^2\psi}{d\xi^2} + \frac{2}{\xi}\frac{d\psi}{d\xi} = -\frac{n}{\xi^2} + \frac{2n}{\xi^2} = \frac{n}{\xi^2}.$$

On the right-hand side:

$$e^{-\psi} = e^{-n\log\xi+\text{const}} = C\cdot\xi^{-n}.$$

To be consistent, both sides must scale the same way, so:

$$\frac{n}{\xi^2} \propto \xi^{-n} \quad \Rightarrow \quad n = 2.$$

Asymptotic Form

At large radii, the solution to the isothermal Lane-Emden equation behaves as:

$$\boxed{\rho(\xi) \propto \xi^{-2}, \qquad \psi(\xi) \sim 2\log\xi + \text{const.}}$$

This is the familiar result for the singular isothermal sphere.

Here's the problem: **this means the mass is infinite**. So what happens if we instead force the system to truncate at a particular radius r_t? In that case, we need to introduce an **external pressure source** in order to make up for the missing gravitational mass. Since $P = K\rho$, we can simply say that

$$P_{\text{ext}} = K\rho(r_T).$$

Stability of BonnorEbert Spheres

A natural question is: **what happens when we perturb the truncation radius r_T slightly**? For instance, a small compression of the sphere might result from infalling material, a shock, or some internal perturbation. The key diagnostic is the response of the required external pressure P_{ext}. If P_{ext} increases to oppose the compression, the configuration is stable; if P_{ext} decreases, the configuration is unstable and the collapse may run away.

We consider a family of truncated isothermal spheres with central density ρ_c, truncated at radius r_T, and held in equilibrium by an external pressure P_{ext}. From the isothermal LaneEmden solution, we define:

- The dimensionless radius: $\xi = r/R_0$, where $R_0^2 = \alpha/(4\pi G\rho_c)$,
- The dimensionless potential: $\psi(\xi)$ such that $\rho(\xi) = \rho_c e^{-\psi(\xi)}$,
- The **mass function** (not the actual mass!):

$$m(\xi) = \xi^2 \frac{d\psi}{d\xi}.$$

Then, the pressure at the outer boundary is

$$P_{\text{ext}} = K\rho(\xi_T) = K\rho_c e^{-\psi(\xi_T)}.$$

We now vary ξ_T slightly and track the behavior of P_{ext}. Taking the derivative:

$$\frac{dP_{\text{ext}}}{d\xi_T} = -K\rho_c e^{-\psi(\xi_T)} \frac{d\psi}{d\xi_T}.$$

This shows that the sign of $dP_{\text{ext}}/d\xi_T$ **depends entirely on the slope $d\psi/d\xi_T$ at the boundary.**

From the LaneEmden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi},$$

we can numerically solve for $\psi(\xi)$ and its derivatives. One then computes $dP_{\text{ext}}/d\xi_T$ and finds that:

- For $\xi_T < \xi_{\text{crit}} \approx 6.451$, the derivative is negative: compressing the sphere increases P_{ext}, which resists the compression. This implies **stability**.
- For $\xi_T > \xi_{\text{crit}}$, the derivative is positive: compressing the sphere decreases P_{ext}, which fails to resist collapse. This implies **instability**.

Thus, the function $P_{\text{ext}}(\xi_T)$ reaches a minimum at $\xi_T = \xi_{\text{crit}}$. Beyond this point, the configuration cannot remain in equilibrium without fine-tuned external support. Thus, the BonnorEbert sphere provides a natural **threshold model for gravitational collapse**.

BonnorEbert Stability Criterion

A BonnorEbert sphere is:

- **Stable** if $\xi_T < \xi_{\text{crit}} \approx 6.451$,
- **Unstable** if $\xi_T > \xi_{\text{crit}}$.

The instability corresponds to collapse under self-gravity once external pressure support becomes insufficient.

7.2.5 Implications of Polytropic Models and BonnorEbert Spheres

Polytropic models provide a powerful, unifying framework for understanding the structure of self-gravitating astrophysical objects under hydrostatic equilibrium. The simplicity of the polytropic equation of state,

$$P = K\rho^{1+1/n},$$

permits analytic and semi-analytic solutions to otherwise intractable stellar structure problems. Their implications span a wide range of astrophysical scenarios:

1. **Unified Treatment of Stellar Interiors**
   Many types of stars from low-mass dwarfs to massive stars are approximately modeled by different polytropic indices. For example:
   - Fully convective stars $\rightarrow n \approx 1.5$,
   - Degenerate cores or white dwarfs $\rightarrow n \approx 1.5$ (non-relativistic) or $n = 3$ (ultra-relativistic),
   - Radiation-pressure dominated massive stars $\rightarrow n \rightarrow 3$.

   This enables comparison across different evolutionary stages and mass regimes.
2. **Insights into Stellar Stability and Limits**
   The solutions to the LaneEmden equation for various n reveal key stability properties. For instance, polytropes with $n < 5$ have finite radius and mass, while those with $n \geq 5$ are unbounded. This connects directly to phenomena like the Chandrasekhar mass limit and the onset of instability in massive stars.
3. **Analytic Scaling Relations**
   Polytropes yield closed-form scaling laws for core pressure, density, temperature, and

radius as functions of mass and composition. These relations are essential for interpreting observations and for initializing stellar evolution models.

4. **BonnorEbert Spheres as Collapse Thresholds**
   The isothermal limit ($n = \infty$) leads to the BonnorEbert sphere: a pressure-confined, self-gravitating equilibrium solution. This model provides a physically motivated criterion for gravitational instability in interstellar gas:
   - For $\xi_T < \xi_{\rm crit} \approx 6.451$, the configuration is stable.
   - For $\xi_T > \xi_{\rm crit}$, small perturbations lead to runaway collapse.

   Observed molecular cloud cores that match the radial density profiles of BonnorEbert spheres can be diagnosed as collapsing or stable depending on their dimensionless radius.
5. **Connection to Star Formation**
   BonnorEbert spheres offer one of the few analytic, observationally testable models for the initial stages of star formation. They predict the critical conditions mass, temperature, external pressure under which a prestellar core transitions from quasi-static equilibrium to gravitational collapse.
6. **Bridge Between Microphysics and Macrophysics**
   Polytropic models illustrate how microphysical assumptions (e.g., ideal gas law, radiation pressure, degeneracy) manifest as macroscopic structure. The mapping from n to physical processes connects thermodynamics, radiative transport, and hydrodynamics in a compact formalism.

7.3 The Intracluster Medium

Galaxy clusters are composed of several mass components, approximately partitioned as:

- **Dark Matter** $\sim 85\%$
- **Hot Ionized Gas (Intracluster Medium)** $\sim 14\%$
- **Stars in Galaxies** $\sim 1\%$

The **intracluster medium (ICM)** is a hot ($T \sim 10^7 10^8$ K), diffuse plasma that emits strongly in the X-ray band via thermal bremsstrahlung and line emission. The X-ray surface brightness and spectral information allow observers to reconstruct the projected temperature and density profiles of the gas.

7.3.1 Hydrostatic Equilibrium in the ICM

For a galaxy cluster in approximate **hydrostatic equilibrium**, the pressure gradient in the ICM must balance the gravitational force from the total mass:

$$\frac{dP}{dr} = -\rho_g(r)\frac{d\Phi}{dr},$$

where:

- $P(r)$ is the gas pressure,
- $\rho_g(r)$ is the gas density,
- $\Phi(r)$ is the total gravitational potential sourced by all mass components: dark matter, stars, and gas.

The pressure of the gas is related to its density and temperature by the ideal gas law:

$$P(r) = \frac{k_B}{\mu m_p}\rho_g(r)T(r) \equiv K\rho_g(r)T(r),$$

where:

- k_B is Boltzmanns constant,
- m_p is the proton mass,
- μ is the mean molecular weight (typically $\mu \approx 0.6$),
- $T(r)$ is the radial temperature profile.

7.3.2 General Expression for the Dynamical Mass

Applying the product rule and dividing by ρ_g, the hydrostatic balance becomes:

$$\frac{1}{\rho_g}\frac{dP}{dr} = K\frac{dT}{dr} + KT\frac{d\log\rho_g}{dr} = -\frac{d\Phi}{dr}.$$

The gravitational acceleration is related to the total enclosed mass $M_{\rm dyn}(<r)$ via:

$$\frac{d\Phi}{dr} = \frac{GM_{\rm dyn}(<r)}{r^2}.$$

Combining these gives:

$$K\left(\frac{dT}{dr} + T\frac{d\log\rho_g}{dr}\right) = -\frac{GM_{\rm dyn}(<r)}{r^2}.$$

Dividing both sides by K and rearranging, we find the key relation for the enclosed dynamical mass:

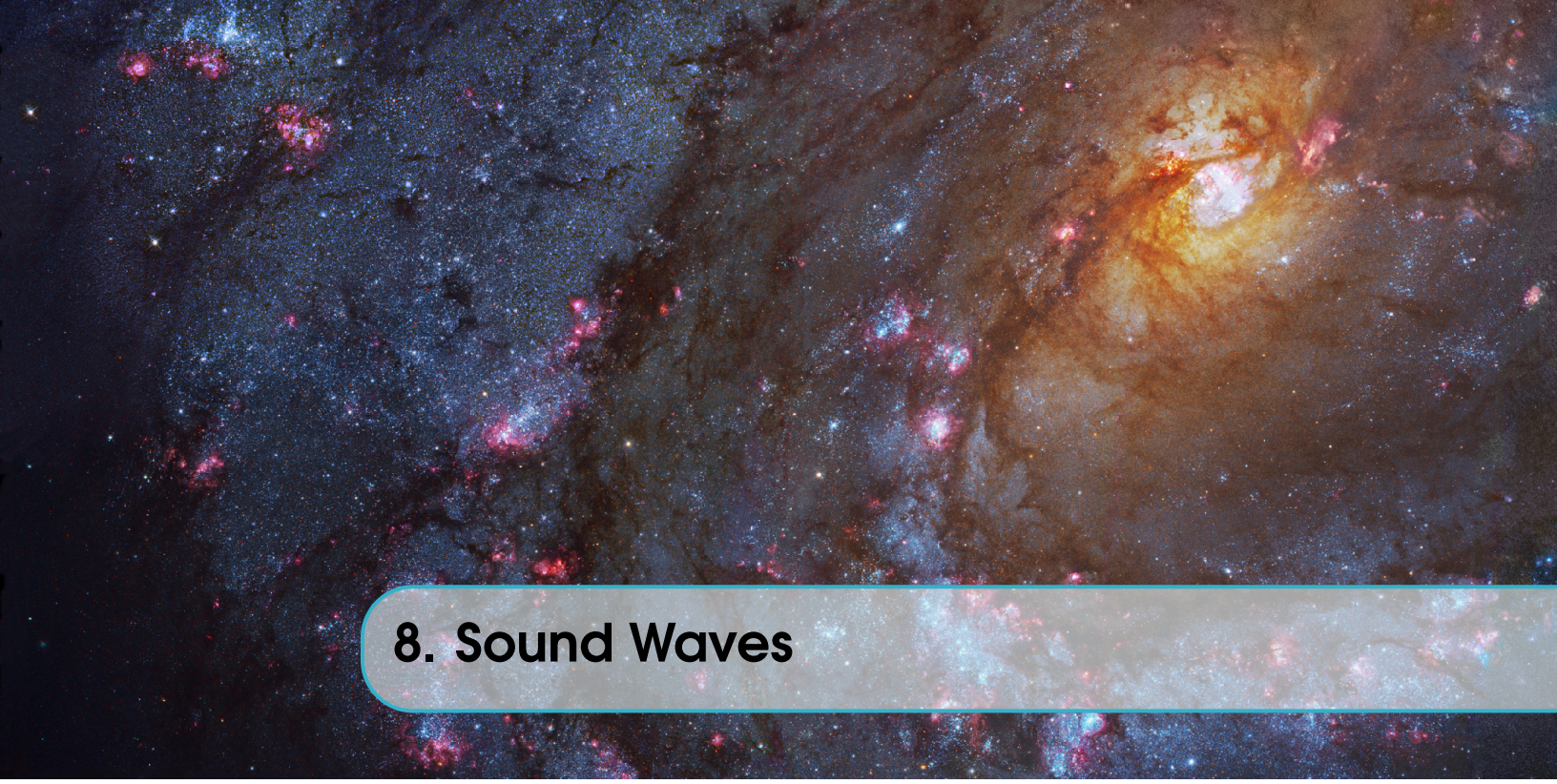
Hydrostatic Mass Equation for the ICM

$$\boxed{M_{\rm dyn}(<r) = -\frac{r^2}{G}\frac{k_BT(r)}{\mu m_p}\left[\frac{d\log T}{dr} + \frac{d\log\rho_g}{dr}\right].} \tag{7.17}$$

This expression allows observers to estimate the total (mostly dark) mass of a cluster using observed profiles of $T(r)$ and $\rho_g(r)$ under the assumption of hydrostatic equilibrium. It is widely used in X-ray and SZ studies of galaxy clusters.

7.3.3 Limitations and Caveats

- The ICM is **not truly isothermal**; real temperature profiles typically decline at large radii.
- Non-thermal pressure support (e.g., turbulence, magnetic fields, cosmic rays) can cause hydrostatic mass estimates to **underestimate** the true mass.
- This method assumes **spherical symmetry** and **equilibrium**, which may be violated in merging or dynamically active clusters.



8. Sound Waves

In this section, we'll discuss the propogation of waves in fluids. In many ways, this is reminiscent of the exercise as it is typically introduced in thermodynamics; however, we will instead formally derive the relevant wave equations and show how the behaviors differ in different scenarios.

This section will primarily utilize **first-order perturbation theory** to explore what happens when we perturb a fluid dynamic system from equilibrium. These perturbations can generate a number of very fascinating phenomena, which, which coupled with various other forces, generate a whole host of wave types relevant in astrophysics:

Table 8.1: Common fluid waves in astrophysical contexts.

Wave Type	**Cause / Restoring Force**	**Astrophysical Relevance**
Sound waves	Pressure gradients (compressibility of gas)	Transport of information and energy; stability of stellar interiors; shocks in ISM and IGM.
Gravity waves (internal)	Buoyancy in stably stratified media	Mixing and transport in stellar interiors; oscillations in neutron stars; atmospheric waves in planets.
Surface gravity waves	Gravity acting at fluid interfaces	Oscillations on stellar surfaces; accretion disk boundaries; starplanet interactions.
Inertial waves	Coriolis force due to rotation	Differential rotation and turbulence in stars and planets; angular momentum transport in disks.
Magnetosonic waves	Combination of pressure gradients and magnetic tension	Propagation of perturbations in magnetized plasmas; shocks in solar wind; structure of jets.
Alfvén waves	Magnetic tension alone	Energy transport in magnetized plasmas; coronal heating; particle acceleration in solar and stellar winds.
KelvinHelmholtz waves	Shear across a fluid interface	Mixing in stellar winds and jets; interface instabilities in accretion flows.

8.1 Elements of Perturbation Theory

The method of perturbations provides a systematic way to study the response of a fluid system to **small deviations from equilibrium**. Rather than attempting to solve the full nonlinear fluid equations, we expand the relevant quantities in small deviations from a

known background solution. Depending on the nature of the perturbation, the system may exhibit one of two broad behaviors:

- **Oscillatory response:** the perturbation remains bounded and typically corresponds to wavelike motion or stable oscillations.
- **Instability:** the perturbation grows with time, often leading to a qualitative change in the system (e.g. turbulence, collapse, or fragmentation).

Formally, we introduce a small parameter $\epsilon \ll 1$ that measures the size of the perturbation relative to the background. If $\psi(x,t)$ is a known equilibrium solution, we expand the perturbed solution as

$$\psi'(x,t) = \psi(x,t) + \epsilon\psi_1(x,t) + \epsilon^2\psi_2(x,t) + \cdots .$$

Substituting into the governing PDE and collecting terms of equal order in ϵ yields a hierarchy of equations. At first order, the **governing equation is linear in ψ_1, justifying the term *linear perturbation theory.***

8.1.1 Fluid Perturbations

As we have discussed in chapter 1, there are two perspectives on fluid dynamics: the **Lagrangian** and the **Eulerian** frameworks. Just as we can describe any **fluid field** ψ in either it's Lagrangian form or its Eulerian form, we can likewise describe perturbations in a field in either an Eulerian formalism or a Lagrangian formalism. As we will see throughout this section, the distinction can be extremely important depending on what sort of problem you seek to solve.

Recall from definitions 1.1.2 and 1.1.3 that a physical quantity ψ (e.g., density, pressure, velocity) may be expressed as either

$$\begin{aligned} \textbf{Eulerian:} \quad & \psi_{\text{Euler}}(\mathbf{x},t) := \psi_{\text{Lagrangian}}(\boldsymbol{\varphi}^{-1}(\mathbf{x},t),t), \\ \textbf{Lagrangian:} \quad & \psi_{\text{Lagrangian}}(\mathbf{X},t) := \psi_{\text{Euler}}(\boldsymbol{\varphi}(\mathbf{X},t),t), \end{aligned}$$

where $\mathbf{X} \in \mathcal{C}_0$ is the material label of a fluid particle (i.e., its position in the reference configuration), and $\boldsymbol{\varphi}$ is the **flow map** that gives the current position $\mathbf{x} \in \mathcal{C}_t$ of that particle at time t.

Let us now consider a fluid in equilibrium $(\rho_0, P_0, \mathbf{u}_0)$. We may introduce a perturbation to these fields in a physical sense; however, just as the underlying fields may be described in either framework, so too can the perturbations:

Definition 8.1.1 — Eulerian perturbation. Let $\psi(\mathbf{x},t)$ be a fluid field (e.g., density, pressure, velocity) and let $\psi_0(\mathbf{x})$ denote its equilibrium value at the same spatial point. The **Eulerian perturbation** is defined by

$$\delta\psi(\mathbf{x},t) := \psi(\mathbf{x},t) - \psi_0(\mathbf{x}).$$

That is, the Eulerian perturbation measures the instantaneous departure of the field from equilibrium at a fixed position in space.

This is, perhaps, the more intuitive of the two: you imagine performing the same fluid flow experiment side by side with marginally different initial conditions. You then

measure the perturbation as the **different between the flows at a given point**.

Definition 8.1.2 — Lagrangian perturbation. Let $\mathbf{X}$ denote the material label of a fluid element in the reference configuration, and let $\boldsymbol{\varphi}(\mathbf{X}, t)$ be the associated flow map that gives the current position $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$.The Lagrangian perturbation of a field is defined as

$$\Delta\psi(\mathbf{X}, t) := \psi(\boldsymbol{\varphi}(\mathbf{X}, t), t) - \psi_0(\mathbf{X}).$$

That is, the Lagrangian perturbation measures the departure of the field experienced by a moving fluid element relative to its equilibrium value.

This is the more difficult quantity to measure. We imagine again running two near identical fluid experiments and labeling the same fluid element in each of the experiments based on the initial configuration. When we run the experiments, the **Lagrangian Perturbation** measures the difference between the **two different fluid elements**.

As such, we really need to assess what we are trying to describe / measure in our computations. The Eulerian perturbation is perhaps easier to measure, but it suffers from the fact that a particular fluid element in one experiment will not generally end up in the same place later on as it does in a second experiment. This is the realm of the Lagrangian perturbation, which is really concerned with the differing experiences of the particles in a given fluid element.

Converting Between Frames

Let's look now at the connection between Eulerian and Lagrangian perturbations. We imagine two fluid experiments: in one, we have a quasi-stable equilibrium described by fluid fields $(\rho_0, P_0, \mathbf{u}_0)$. In the other, we have the same system subjected to a perturbation:

$$(\rho_0, P_0, \mathbf{u}_0) \to (\rho_0 + \delta\rho, P_0 + \delta P, \mathbf{u}_0 + \delta\mathbf{u}).$$

In the **unperturbed flow**, a particular fluid element $\mathbf{X}$ (Lagrangian frame) will move through space as determined by the **flow map** $\boldsymbol{\varphi}(\mathbf{X}, t)$. We can describe how far that fluid element has gone by

$$\boldsymbol{\xi}(\mathbf{X}, t) = \boldsymbol{\varphi}(\mathbf{X}, t) - \boldsymbol{\varphi}(\mathbf{X}, 0) = \boldsymbol{\varphi}(\mathbf{X}, t) - \mathbf{X}.$$

This is the so-called **displacement field**: it tells us how far away a particular fluid element is from its starting position. Let's consider the perturbed flow. Clearly there will be a slightly different flow map $\boldsymbol{\vartheta}$ and subsequently a slightly different displacement field $\boldsymbol{\varrho}$ such that

$$\boldsymbol{\varrho}(\mathbf{X}, t) = \boldsymbol{\vartheta}(\mathbf{X}, t) - \mathbf{X}.$$

Let's now ask the following question: **what are the perturbations we measure?** Consider a field ψ of the flow. For a particular position in space and time $(\mathbf{X}, t)$, the **Eulerian perturbation** will be

$$\delta\psi(\mathbf{x}, t) = \psi(\mathbf{x}, t) - \psi_0(\mathbf{x}, t).$$

What happens to the **Lagrangian Perturbation**? Well that is

$$\Delta\psi(\mathbf{X},t) = \underbrace{\psi(\boldsymbol{\vartheta}(\mathbf{X},t),t)}_{\text{New field @ new location}} - \underbrace{\psi_0(\boldsymbol{\varphi}(\mathbf{X},t),t)}_{\text{Old field @ old location}}.$$

We can make the relation between $\delta\psi$ and $\Delta\psi$ precise by Taylor-expanding the perturbed flow about the unperturbed one. Write

$$\boldsymbol{\vartheta}(\mathbf{X},t) = \boldsymbol{\varphi}(\mathbf{X},t) + \boldsymbol{\eta}(\mathbf{X},t),$$

where

$$\boldsymbol{\eta}(\mathbf{X},t) := \boldsymbol{\varrho}(\mathbf{X},t) - \boldsymbol{\xi}(\mathbf{X},t)$$

is the *perturbative displacement*: the difference between the two trajectories for the same material element $\mathbf{X}$.

Substituting into the Lagrangian perturbation,

$$\Delta\psi(\mathbf{X},t) = \psi(\boldsymbol{\varphi}(\mathbf{X},t) + \boldsymbol{\eta}(\mathbf{X},t),t) - \psi_0(\boldsymbol{\varphi}(\mathbf{X},t),t).$$

Now expand the first term about $\boldsymbol{\varphi}(\mathbf{X},t)$:

$$\psi(\boldsymbol{\varphi}+\boldsymbol{\eta},t) = \psi(\boldsymbol{\varphi},t) + \boldsymbol{\eta}\cdot\nabla\psi(\boldsymbol{\varphi},t) + \mathcal{O}(|\boldsymbol{\eta}|^2).$$

Hence,

$$\Delta\psi(\mathbf{X},t) = \underbrace{\big[\psi(\boldsymbol{\varphi},t) - \psi_0(\boldsymbol{\varphi},t)\big]}_{\delta\psi(\boldsymbol{\varphi},t)} + \boldsymbol{\eta}\cdot\nabla\psi_0(\boldsymbol{\varphi},t) + \mathcal{O}(|\boldsymbol{\eta}|^2).$$

To first order, we therefore obtain the fundamental relation

$$\boxed{\Delta\psi = \delta\psi + \boldsymbol{\eta}\cdot\nabla\psi_0}. \tag{8.1}$$

R For those well versed in differential geometry, this is precisely the **Lie Derivative**. Additionally, we should mention that for a static background $\boldsymbol{\eta} = \boldsymbol{\varrho}$.

Big Idea

There are two ways to compare perturbed and unperturbed flows:

- In the **Eulerian frame**, we stay at the same point in space and measure how the field changes there:

$$\delta\psi(\mathbf{x},t) = \psi(\mathbf{x},t) - \psi_0(\mathbf{x},t).$$

- In the **Lagrangian frame**, we follow a given fluid element along its trajectory and compare what it experiences in the perturbed vs. unperturbed flow:

$$\Delta\psi(\mathbf{X},t) = \psi(\boldsymbol{\vartheta}(\mathbf{X},t),t) - \psi_0(\boldsymbol{\varphi}(\mathbf{X},t),t).$$

The key difference is thus: *Eulerian perturbations compare fields at the same location, while Lagrangian perturbations compare fields along the same material elements path.*

$$\boxed{\Delta\psi \;=\; \delta\psi \;+\; \boldsymbol{\eta}\cdot\nabla\psi_0}$$

Here $\boldsymbol{\eta}$ is the perturbative displacement (the difference between the perturbed and unperturbed trajectories). This conversion formula is the workhorse for translating between the two perspectives.

8.2 The General Approach

Fluids possess two key physical properties: they can *transmit momentum* through internal stresses, and they can *oscillate about an equilibrium configuration* when displaced. These features make them natural hosts for a wide variety of wave phenomena. In fluid dynamics, the systematic study of such phenomena is most effectively carried out using **first-order perturbation theory**, in which small deviations from an established equilibrium state are introduced and their subsequent evolution is analyzed.

The Equilibrium Configuration

The first step in any perturbative analysis is to specify the equilibrium state of the system. In the present context, this corresponds to a hydrostatic configuration in which the velocity field vanishes, $\mathbf{u} = 0$, and the pressure and density take prescribed background forms, $p = p_0(\mathbf{x})$ and $\rho = \rho_0(\mathbf{x})$. All subsequent perturbations will be defined relative to this reference state.

R As we will see, the spatial structure of the equilibrium configuration can strongly influence both the types of waves that can exist and the manner in which they propagate through the medium.

The Perturbation

With the equilibrium configuration specified, we are able to introduce a perturbation to that original configuration. One can choose to either view that perturbation in the Lagrangian framework or the Eulerian one (*it is important to remember that ANY perturbation is BOTH Eulerian and Lagrangian depending on how you choose to view it*). For need of a choice, we'll use the **Eulerian** (δ) perturbation here, but as will be explored below, the relevant equations may be derived from either framework. Thus, our perturbed fields introduce a new **initial boundary value problem (IBVP)** with initial conditions

$$\begin{aligned}\rho(\mathbf{x},0) &= \rho_0(\mathbf{x},0) + \delta\rho(\mathbf{x},0)\\ P(\mathbf{x},0) &= P_0(\mathbf{x},0) + \delta P(\mathbf{x},0)\\ \mathbf{u}(\mathbf{x},0) &= \delta\mathbf{u}(\mathbf{x},0).\end{aligned} \tag{8.2}$$

Of course, introducing this perturbation will also change the flow map $\boldsymbol{\varphi}$ from the Lagrangian framework and the corresponding displacement fields.

Our goal now is to determine the equations of motion for the flow in terms of the first order perturbations and solve them using linearization.

8.2.1 The Linearized Fluid Equations

As is the standard approach in all forms of perturbation theory, we now consider what happens to the **continuity equation** and the **Euler equation** under the assumption of linear perturbation. Let's look first at the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Letting $\rho \mapsto \rho_0 + \delta\rho$, etc. we have

$$\underbrace{\frac{\partial \rho_0}{\partial t}}_{0 \text{ by HSE}} + \frac{\partial \delta\rho}{\partial t} + \nabla \cdot \left(\rho_0 \delta\mathbf{u} + \underbrace{\delta\rho\delta\mathbf{u}}_{\text{2nd order}} \right) = 0,$$

which takes the linearized form

$$\boxed{\frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\rho_0 \delta\mathbf{u}) = 0.} \tag{8.3}$$

In the **Lagrangian** approach, inserting our perturbation (*In the LAGRANGIAN form*),

$$\frac{D(\rho_0 + \Delta\rho)}{Dt} + (\rho_0 + \Delta\rho)\nabla \cdot \mathbf{u} = 0.$$

Keeping **only first order terms** and recognizing that $\mathbf{u} = \partial_t \boldsymbol{\xi}$,

$$\boxed{\frac{D\Delta\rho}{Dt} + \rho_0 \nabla \cdot \partial_t \boldsymbol{\xi} = 0.} \tag{8.4}$$

We now look at the **Euler Equation**. Assuming the situation to be inviscid, we have

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \rho \mathbf{f}_{\text{ext}}.$$

In the context of our perturbative expansion, we can eliminate higher order terms to find

$$\rho_0 \frac{\partial \delta\mathbf{u}}{\partial t} = \underbrace{-\nabla P_0 + \rho_0 \mathbf{f}_{\text{ext}}}_{=0 \text{ by HSE}} - \nabla \delta P + \delta\rho \mathbf{f}_{\text{ext}}.$$

Thus, the **linearized Euler's Equation** is

$$\boxed{\frac{\partial \delta\mathbf{u}}{\partial t} = -\frac{\nabla \delta P}{\rho_0} + \frac{\delta\rho}{\rho_0}\mathbf{f}_{\text{ext}}.} \tag{8.5}$$

In the Lagrangian framework,

$$\frac{D\Delta\mathbf{u}}{Dt} = \frac{1}{\rho_0}\left[\Delta\rho \mathbf{f}_{\text{ext}} - \nabla \Delta P\right].$$

Notably, remember that $\mathbf{u} = \Delta\mathbf{u} = \partial_t \boldsymbol{\xi}$. Additionally, $\partial_t \boldsymbol{\xi} \sim D_t \boldsymbol{\xi}$ to first order. Thus,

$$\boxed{\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \frac{1}{\rho_0}\left[\Delta\rho \mathbf{f}_{\text{ext}} - \nabla \Delta P\right]} \tag{8.6}$$

With these, we now have the fully linearized equations for our fluid flow problems.

8.2.2 The Wave Equation

We now derive the wave equation for small perturbations of a barotropic fluid, working in the Lagrangian formalism introduced above and then in the Eulerian formalism.

The Lagrangian Wave Equation

Starting from the linearized Euler and continuity equations in Lagrangian form, the structure of a wave equation quickly emerges. Taking the divergence of the linearized Euler equation gives

$$\nabla \cdot \left(\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \right) = \rho_0 \nabla \cdot \left(\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \right) + \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \cdot \nabla \rho_0 = -\nabla^2 \Delta p + \nabla \cdot (\Delta \rho \, \mathbf{f}_{\text{ext}}).$$

Meanwhile, taking the material derivative of the linearized continuity equation yields

$$\frac{\partial^2 \Delta \rho}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = 0.$$

Combining these two results eliminates the divergence of $\partial_t^2 \boldsymbol{\xi}$:

$$\frac{\partial^2 \Delta \rho}{\partial t^2} - \nabla^2 \Delta p + \nabla \cdot (\Delta \rho \, \mathbf{f}_{\text{ext}}) = \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \cdot \nabla \rho_0.$$

For a barotropic fluid, $p = p(\rho)$, so that

$$dp = \frac{\partial p}{\partial \rho} \, d\rho,$$

and therefore

$$\Delta p = c_s^2 \, \Delta \rho, \qquad c_s^2 \equiv \frac{\partial p}{\partial \rho}.$$

Substituting, the equation becomes

$$\frac{\partial^2 \Delta \rho}{\partial t^2} - \nabla^2 \left(c_s^2 \, \Delta \rho \right) = \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \cdot \nabla \rho_0 - \nabla \cdot (\Delta \rho \, \mathbf{f}_{\text{ext}}).$$

Finally, using the Euler equation in Lagrangian form,

$$\frac{D^2 \boldsymbol{\xi}}{Dt^2} = \partial_t^2 \boldsymbol{\xi} = -\frac{\nabla \Delta p}{\rho_0} + \frac{\Delta \rho}{\rho_0} \, \mathbf{f}_{\text{ext}},$$

we may write the full wave equation as

$$\underbrace{\frac{\partial^2 \Delta \rho}{\partial t^2} - \nabla^2 \left(c_s^2 \Delta \rho \right)}_{\text{Classical wave operator}} = \underbrace{-\frac{\nabla \rho_0}{\rho_0} \cdot \left[\nabla \left(c_s^2 \Delta \rho \right) - \Delta \rho \, \mathbf{f}_{\text{ext}} \right]}_{\text{Background coupling}} - \underbrace{\nabla \cdot (\Delta \rho \, \mathbf{f}_{\text{ext}})}_{\text{Buoyancy/external forcing}} \tag{8.7}$$

The Eulerian Wave Equation

We can obtain an analogous wave equation directly in the Eulerian framework. Starting from the linearized continuity equation (8.3),

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot \big(\rho_0 \delta\mathbf{u}\big) = 0,$$

and the linearized Euler equation (8.5),

$$\frac{\partial \delta\mathbf{u}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} + \frac{\delta\rho}{\rho_0}\,\mathbf{f}_{\text{ext}},$$

we may eliminate $\delta\mathbf{u}$ to obtain a closed equation for $\delta\rho$. Taking another time derivative of the continuity equation, one finds that

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \nabla \cdot \left(\rho_0 \frac{\partial \delta\mathbf{u}}{\partial t}\right) = 0.$$

Substituting for $\partial_t \delta\mathbf{u}$ in the Euler equation gives

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \nabla \cdot (-\nabla \delta p + \delta\rho\,\mathbf{f}_{\text{ext}}) = 0.$$

For a barotropic equation of state,

$$\delta p = c_s^2 \delta\rho, \qquad c_s^2 \equiv \frac{\partial p}{\partial \rho}.$$

Hence,

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2\big(c_s^2 \delta\rho\big) + \nabla \cdot (\delta\rho\,\mathbf{f}_{\text{ext}}) = 0.$$

$$\boxed{\underbrace{\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2\big(c_s^2 \delta\rho\big)}_{\text{Classical wave operator}} + \underbrace{\nabla \cdot (\delta\rho\,\mathbf{f}_{\text{ext}})}_{\text{Buoyancy/external forcing}} = 0} \tag{8.8}$$

The Wave Equation: Qualitatively

Both the Lagrangian (8.7) and Eulerian (8.8) formulations describe the same underlying physics: the oscillatory response of a compressible fluid to small perturbations. However, the way in which background structure and external forcing appear in the equations differs, and this difference shapes how we interpret the dynamics.

In the Lagrangian description, we follow individual fluid elements through time. The central variable is the *displacement field* $\boldsymbol{\xi}$, and the resulting wave equation for the density perturbation is

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - \nabla^2\big(c_s^2 \Delta\rho\big) = -\frac{\nabla\rho_0}{\rho_0} \cdot \Big[\nabla\big(c_s^2 \Delta\rho\big) - \Delta\rho\,\mathbf{f}_{\text{ext}}\Big] - \nabla \cdot (\Delta\rho\,\mathbf{f}_{\text{ext}})\,.$$

The left-hand side looks like a classical wave operator (time derivatives opposed by a pressure-gradient restoring force). The right-hand side, however, makes **explicit** the ways in which spatial gradients in the background ($\nabla\rho_0$) and external forces ($\mathbf{f}_{\text{ext}}$) couple to the oscillations. T**his makes the Lagrangian framework particularly transparent for studying buoyancy, stratification, and oscillation modes in stars and planets.**

In the Eulerian description, we monitor perturbations at fixed spatial points. The governing equation for the density perturbation takes the simpler form

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2\left(c_s^2 \delta\rho\right) + \nabla\cdot(\delta\rho\,\mathbf{f}_{\text{ext}}) = 0.$$

Here the background gradient terms do not appear explicitly. Instead, the effects of stratification and equilibrium structure are *hidden inside* the relationship between Eulerian and Lagrangian perturbations:

$$\Delta\rho = \delta\rho + \boldsymbol{\xi}\cdot\nabla\rho_0.$$

This makes the Eulerian equation more compact and directly suited for numerical simulation, but less explicit about the underlying physical couplings.

Big Idea

- **Lagrangian form (follows fluid elements):**

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - \nabla^2\left(c_s^2 \Delta\rho\right) = -\frac{\nabla\rho_0}{\rho_0}\cdot\left[\nabla\left(c_s^2\Delta\rho\right) - \Delta\rho\,\mathbf{f}_{\text{ext}}\right] - \nabla\cdot(\Delta\rho\,\mathbf{f}_{\text{ext}})\,.$$

- **Eulerian form (follows spatial points):**

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2\left(c_s^2 \delta\rho\right) + \nabla\cdot(\delta\rho\,\mathbf{f}_{\text{ext}}) = 0.$$

They are related by the conversion formula

$$\Delta\rho = \delta\rho + \boldsymbol{\xi}\cdot\nabla\rho_0,$$

which bridges the two perspectives.

8.3 The Speed of Sound

In the discussion above, we introduced the sound speed c_s as the derivative

$$c_s^2 = \frac{\partial p}{\partial \rho},$$

which acts as the effective restoring force for compressional perturbations. However, in practice, there are multiple definitions of the sound speed depending on the thermodynamic assumptions we adopt. Most commonly, we distinguish between the **isothermal sound speed** and the **adiabatic sound speed**.

The Isothermal Sound Speed

For an ideal gas under strictly isothermal conditions (temperature held fixed), the sound speed is

$$c_{\rm iso} = \sqrt{\frac{k_B T}{\mu m_p}}, \tag{8.9}$$

where k_B is the Boltzmann constant, T is the temperature, μ is the mean molecular weight, and m_p is the proton mass. This form is appropriate when heat exchange with the surroundings is highly efficient, so that perturbations occur at effectively constant temperature.

The Adiabatic Sound Speed

By contrast, when heat exchange is inefficient and perturbations evolve at constant entropy, the relevant speed is the *adiabatic* sound speed:

$$c_{\rm ad} = \sqrt{\frac{\gamma k_B T}{\mu m_p}} = \sqrt{\frac{\gamma P}{\rho}}, \tag{8.10}$$

where γ is the adiabatic index of the gas. Here the restoring force is stronger because compression not only increases the density but also heats the gas, producing a larger pressure response.

Choosing the Correct Sound Speed

The distinction between $c_{\rm iso}$ and $c_{\rm ad}$ reflects the **efficiency of thermal exchange relative to the oscillation timescale**. In the rapid-exchange (isothermal) limit, the fluid remains at constant temperature, while in the slow-exchange (adiabatic) limit, the fluid retains its entropy. In astrophysical applications one must take care: e.g., sound waves in stellar interiors are typically adiabatic, while waves in interstellar gas clouds may be closer to isothermal, depending on cooling timescales.

Big Idea

- **Isothermal:**

  $$c_{\rm iso} = \sqrt{\frac{k_B T}{\mu m_p}},$$

  valid when heat exchange is rapid.
- **Adiabatic:**

  $$c_{\rm ad} = \sqrt{\frac{\gamma k_B T}{\mu m_p}} = \sqrt{\frac{\gamma P}{\rho}},$$

  valid when heat exchange is negligible.

8.4 Free Waves in Uniform Media

The uniform medium is the simplest setting in which to study fluid waves. Here both the equilibrium density ρ_0 and pressure p_0 are spatially constant, and we neglect any external forces. In this case, the wave equation reduces to the particularly transparent form

$$\partial_t^2 \Delta\rho - c_s^2 \nabla^2 \Delta\rho = 0,$$

which is nothing other than the **classical wave equation**. A natural way to solve such an equation is to consider plane-wave disturbances,

$$\Delta\rho(\mathbf{x}, t) = \Delta\rho_0 \, \exp(i \, [\mathbf{k} \cdot \mathbf{x} - \omega t]) \, .$$

Substituting this ansatz gives the **dispersion relation**

$$\omega^2 = c_s^2 |\mathbf{k}|^2.$$

The immediate consequences are striking:

- The **wave speed** is exactly c_s, the adiabatic sound speed.
- The **phase velocity** and **group velocity** are both c_s.
- The wave is therefore **non-dispersive**: all Fourier components propagate without distortion.

Relationships Between Fields

Because $p = p(\rho)$ for a barotropic fluid, the pressure and density perturbations are in phase:

$$\Delta p = c_s^2 \Delta\rho.$$

Likewise, from the linearized continuity equation

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{u} = 0,$$

the plane-wave form yields

$$\mathbf{k} \cdot \delta\mathbf{u} = \frac{\omega}{\rho_0} \, \delta\rho.$$

This tells us that the velocity perturbation is **parallel to the wavevector k**: sound waves in fluids are inherently **longitudinal waves**.

This uniform case provides a baseline for understanding real astrophysical contexts. In stellar interiors, for instance, acoustic waves probe temperature and density profiles; in the interstellar medium, sound waves mediate the propagation of turbulence and shocks. Of course, true astrophysical environments are rarely uniform: stratification, gravity, rotation, and magnetic fields all enrich the dynamics and lead to new wave families. But the uniform, barotropic sound wave remains the essential prototype.

Big Idea

Acoustic plane waves in a uniform medium.
For a plane wave with wavevector $\mathbf{k}$ and frequency $\omega = c_s|\mathbf{k}|$, the first-order perturbations take the explicit form

$$\begin{aligned}\Delta\rho(\mathbf{x},t) &= \Delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},\\ \Delta p(\mathbf{x},t) &= c_s^2\Delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},\\ \delta\mathbf{u}(\mathbf{x},t) &= \frac{\omega}{\rho_0|\mathbf{k}|^2}\,\mathbf{k}\,\Delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}.\end{aligned}$$

- Density and pressure oscillations are *in phase*.
- The velocity perturbation points *along the wavevector*: the wave is longitudinal.
- All fields oscillate coherently and propagate at the sound speed c_s.

8.5 Waves in Stratified Media

As we have discussed in previous chapters, most astrophysical fluids are not uniform media, but are instead **stratified**. The most famous of these is, of course, the material making up a star. In these systems, the **background coupling** of the wave leads to interesting behaviors which are worthy of considerably theoretical investment. In this section, we'll discuss the details of these sorts of waves.

8.5.1 Surface Waves (Water Waves)

We consider small-amplitude waves on an **incompressible, inviscid, irrotational fluid** of constant density ρ_0, with a flat bed at $z = -H$ and a free surface $z = \eta(x,t)$ exposed to quiescent air of pressure P_{atm}. Gravity acts as $\mathbf{f}_{\text{ext}} = -g\,\hat{\mathbf{z}}$. In this scenario, we will see a few critical physical principles at play: first off, we will see how the incompressibility assumption leads to interesting mathematical structure. We will also, for the first time, see some of the complications which arise at boundaries between flows.

Equilibrium Configuration

As in **most wave problems**, we presume a hydrostatic ambient environment with pressure balanced against the force of gravity. The **Euler Equation** is

$$\frac{\nabla P_0}{\rho_0} = -g\,\hat{\mathbf{z}}.$$

As such,

$$\boxed{\nabla P_0 = -\rho_0 g\hat{\mathbf{z}} \;\Rightarrow\; P_0(z) = P_{\text{atm}} - \rho_0 g z, \qquad P_0(0) = P_{\text{atm}}.} \tag{8.11}$$

It is worthwhile to remember (as it will arise later) that this solution is actually valid for ***any*** *z so long as there is fluid at that point. We* ***have not assumed anything about the surface*** *in doing this computation.*

Perturbative Analysis

As usual we now proceed by introducing the perturbations. We have **incompressibility**, so we cannot perturb the density. We therefore perturb just P ($P_0 \to P_0 + \delta P$), and $\mathbf{u}$ ($\mathbf{u} = \delta\mathbf{u}$). Incompressibility and irrotationality imply the existence of potential flow mediated by a potential ψ.

$$\nabla \cdot \delta\mathbf{u} = 0, \qquad \nabla \times \delta\mathbf{u} = 0 \;\Rightarrow\; \delta\mathbf{u} = \nabla\psi, \qquad \nabla^2\psi = 0 \quad \text{in} \quad -H < z < 0. \tag{8.12}$$

We have therefore our differential equation; however, we **need to enforce sufficient boundary conditions** to ensure that our problem is well posed. We have two boundaries: the bottom of the flow and the top of the flow. The impermeable bed enforces the bottom boundary condition

$$\partial_z\psi(x, -H, t) = 0. \tag{8.13}$$

Across the interface at the top of the fluid, the jump in traction balances any surface stresses:

$$\left(\boldsymbol{\sigma}^{\text{fluid}} - \boldsymbol{\sigma}^{\text{air}}\right) \cdot \mathbf{n} = \nabla_s \cdot \boldsymbol{\tau}_s. \tag{8.14}$$

The argument for this is that the boundary has infinitesmal mass, which means that there cannot be a net force on it which is not accounted for internally. **For inviscid media with no surface tension** ($\boldsymbol{\tau}_s = \mathbf{0}$) and isotropic stress $\boldsymbol{\sigma} = -P\,\mathbf{I}$,

$$\left(-P_{\text{fluid}} + P_{\text{air}}\right)\mathbf{n} = \mathbf{0} \;\Rightarrow\; P_{\text{fluid}} = P_{\text{air}} \quad \text{at } z = \eta(x, t). \tag{8.15}$$

Our conclusion is therefore that the **pressure is continuous at the boundary**. This has implications for what the pressure does at $z = 0$, which is now *not necessarily at the surface.* To first order, we recognize that the **unperturbed** pressure at η was

$$P_0(x, z) = P_{\text{atm}} - \rho_0 g z \implies P_0(x, \eta) = P_{\text{atm}} - \rho_0 g \eta.$$

Now, in the **perturbed scenario**, we have $P(x, \eta) = P_{\text{atm}}$, so we recognize that our perturbation must be

$$\delta P = \rho_0 g \eta.$$

R A subtle point arises here: in equilibrium the fluid only occupies $-H \le z \le 0$, so it might seem questionable to evaluate the background hydrostatic profile $P_0(z)$ at $z = \eta > 0$, where no fluid existed initially. The correct interpretation is as follows. The boundary condition always requires

$$P(x, \eta, t) = P_{\text{atm}},$$

at the *actual* perturbed free surface. Writing $P = P_0 + \delta P$ and expanding the analytic background profile $P_0(z)$ about $z = 0$, we have

$$P_0(\eta) \approx P_0(0) + \eta\,\partial_z P_0(0).$$

This expansion should not be understood as asserting that the equilibrium fluid extended into $z > 0$; rather, it is a linearization of the boundary condition about the reference surface $z = 0$, using the known gradient of the hydrostatic state. In this way the pressure perturbation at $z = 0$ is tied directly to the surface displacement, without requiring any physical continuation of the equilibrium fluid above $z = 0$.

From the linearized Euler equation for potential flow (equivalently, the linear unsteady Bernoulli relation),

$$\frac{\partial \nabla\phi}{\partial t} = -\frac{\nabla P}{\rho_0} \implies \delta P = -\rho_0\, \partial_t \psi, \tag{8.16}$$

so our condition becomes the *dynamic free-surface condition*

$$\partial_t \psi(x, 0, t) = g\, \eta(x, t). \tag{8.17}$$

We have a final boundary condition to enforce with regard to the surface. The free surface $z = \eta(x, t)$ is a *material surface*: each parcel of fluid that lies on the interface at some time must remain on the interface as it moves. Physically, this just says that the interface cannot detach from the fluid or slip past itthe surface velocity is exactly the fluid velocity at that location. *This is an **assertion**, not a god given fact. The idea is that we should not allow a particle of fluid to advect across the boundary because this could create pockets of fluid or other issues. Instead we require that a particle on the surface always stays on the surface.* Mathematically, if we define the surface function

$$F(x, z, t) := z - \eta(x, t),$$

then the condition that fluid parcels remain on the surface is expressed by the vanishing of its material derivative, *since its advected with the flow*

$$\frac{DF}{Dt} = \partial_t F + \delta\mathbf{u} \cdot \nabla F = 0.$$

Since $\nabla F = \hat{\mathbf{z}} - \partial_x \eta\, \hat{\mathbf{x}}$, this yields the exact kinematic boundary condition

$$\partial_t \eta = \delta\mathbf{u}_z - \delta\mathbf{u}_x \cdot \nabla_x \eta, \tag{8.18}$$

For small-amplitude waves, the surface slopes are small ($|\nabla_\parallel \eta| \ll 1$), so the nonlinear advection term may be neglected. The kinematic boundary condition then linearizes to

$$\partial_t \eta(x, t) = \delta\mathbf{u}_z = \partial_z \psi(x, 0, t), \tag{8.19}$$

where we used $\delta\mathbf{u} = \nabla\psi$.

Structure of Waves

Let's summarize our findings so far. We determined that these surface waves satisfy Laplace's Equation, and that they are subjected to three different boundary conditions:

1. **Impermeability at the bottom**: Leads to $\partial_z \psi(x, -H) = 0$.
2. **Surface Forces**: Lead to the condition that $\partial_t \psi(x, 0, t) = g\, \eta(x, t)$.
3. **Material Surface**: Requires that $\partial_z \psi(x, 0, t) = \partial_t \eta$.

We now look to find a valid solution. Seeking $x$propagating normal modes, take

$$\psi(x, z, t) = A\, \cosh\big(k(z + H)\big)\, e^{i(kx - \omega t)}, \qquad \eta(x, t) = \eta_0\, e^{i(kx - \omega t)}. \tag{8.20}$$

This satisfies (8.12) and (8.13). *The really thorough approach is to perform separation of variables, identify the exponential and sinusoidal components, and then proceed from there; however, this is what you would find.* Apply the surface boundary conditions:

Kinematic BC (8.19)**.**

$$-i\omega\eta_0 = \partial_z\psi|_{z=0} = Ak\sinh(kH) \quad \Rightarrow \quad A = -\frac{i\omega}{k\,\sinh(kH)}\,\eta_0. \tag{8.21}$$

Dynamic BC (8.17)**.**

$$\partial_t\psi|_{z=0} = -i\omega\,A\cosh(kH) = g\,\eta_0. \tag{8.22}$$

Insert (8.21) into (8.22):

$$-i\omega\left(-\frac{i\omega}{k\,\sinh(kH)}\,\eta_0\right)\cosh(kH) = g\,\eta_0 \quad \Rightarrow \quad \frac{\omega^2}{k}\,\coth(kH) = g. \tag{8.23}$$

Therefore, the *gravity-wave dispersion relation* at finite depth is

$$\boxed{\omega^2 = gk\,\tanh(kH)\,.} \tag{8.24}$$

The Dispersion Relation

We have derived that smallamplitude surface gravity waves on a fluid of finite depth obey the dispersion relation

$$\omega^2 = gk\tanh(kH).$$

This compact formula encodes a great deal of physical information. In particular, it bridges smoothly between **the two limiting regimes of *deep water* and *shallow water***, while providing corrections in the intermediate case. Let us examine these limits carefully.

Deepwater limit ($kH \gg 1$**).**

For very shortwavelength waves relative to the depth of the fluid, the bottom plays essentially no role. In this limit

$$\tanh(kH) \;\to\; 1,$$

so the dispersion reduces to

$$\omega^2 \;=\; gk. \tag{8.25}$$

The phase speed $c_{\rm ph} = \omega/k$ and group speed $c_{\rm g} = d\omega/dk$ are therefore

$$c_{\rm ph} = \sqrt{\frac{g}{k}}, \qquad c_{\rm g} = \tfrac{1}{2}\,c_{\rm ph}.$$

Thus, in deep water, **longer waves travel faster, and wave packets disperse strongly because the group speed is only half of the phase speed.** This is why in the ocean swell patterns tend to sort themselves by wavelength: long swells outrun the shorter ripples.

Shallowwater limit ($kH \ll 1$).
For very longwavelength waves compared to the depth, the hyperbolic tangent can be approximated by

$$\tanh(kH) \;\approx\; kH.$$

The dispersion then becomes

$$\omega^2 \;\approx\; gH\,k^2. \tag{8.26}$$

In this case the phase and group speeds are

$$c_{\mathrm{ph}} = \sqrt{gH}, \qquad c_{\mathrm{g}} = \sqrt{gH}.$$

That is, both speeds are independent of wavelength: **shallowwater gravity waves are *nondispersive.*** All wavelengths travel at the same speed, determined only by the depth of the fluid.

Intermediate regime.
Between these extremes ($kH \sim 1$), the full dispersion relation

$$\omega^2 = gk\,\tanh(kH)$$

must be used. In this case the waves are only partially dispersive: shorter wavelengths feel the influence of the bottom less strongly, while longer wavelengths are more affected. The phase speed interpolates smoothly between $\sqrt{g/k}$ in deep water and $\sqrt{gH}$ in shallow water. It is often convenient to summarize the behavior by expanding the hyperbolic tangent. For small but finite kH,

$$\tanh(kH) \;\approx\; kH - \tfrac{1}{3}(kH)^3 + \cdots,$$

so that

$$\omega^2 \;\approx\; gH\,k^2\left[1 - \tfrac{1}{3}(kH)^2 + \cdots\right].$$

This provides the leadingorder dispersive correction to the shallowwater limit.

8.5.2 Surface Tension and Capillary Waves

In surface waves with very short wavelengths, the effect of **surface tension** becomes significant. Physically, surface tension arises from the cohesive molecular forces within the liquid being stronger than those across the liquid–air boundary. Molecules at the interface are therefore pulled inward, giving the free surface an effective "membrane tension" that resists deformations.

Definition 8.5.1 — Surface Tension. The *surface tension* γ is defined as the energy per unit area required to create new surface,

$$\gamma = \frac{dE}{dA},$$

with units of N/m (force per unit length) or J/m^2 (energy per unit area). For isotropic interfaces, γ is a constant independent of direction along the surface. In a more general

scenario, we introduce the so-called **surface stress tensor** defined as

$$\boldsymbol{\tau}_s = \gamma\, \mathbf{I}_s,$$

for an isotropic interface where I_s is the projection onto the tangent plane: $I_s = I - \mathbf{n}\otimes\mathbf{n}$.

This definition highlights that surface tension acts as a **restoring mechanism** for perturbations of the surface: deforming the interface increases its area, which costs energy.

The more formal way to incorporate surface tension into the equations of motion is through a *surface stress tensor* $\boldsymbol{\tau}_s$. The general traction balance across the interface is

$$\left(\boldsymbol{\sigma}^{\text{fluid}} - \boldsymbol{\sigma}^{\text{air}}\right)\cdot \mathbf{n} = \nabla_s \cdot \boldsymbol{\tau}_s, \tag{8.27}$$

where ∇_s denotes the surface divergence operator intrinsic to the interface. Taking the surface divergence yields

$$\nabla_s \cdot \boldsymbol{\tau}_s = -\gamma\, \kappa\, \mathbf{n},$$

where κ is the *mean curvature* of the surface. This reproduces the celebrated *Laplace–Young condition*:

$$P_{\text{fluid}} - P_{\text{air}} = \gamma\, \kappa. \tag{8.28}$$

R This relation shows directly how the **curvature couples to surface tension**. The intuitive reason is that deforming a curved interface changes its area linearly in proportion to the mean curvature, so the variational derivative of the surface energy produces a restoring force proportional to κ. Flat interfaces ($\kappa = 0$) therefore feel no capillary restoring force, while highly curved interfaces resist deformation strongly.

Linearized Form for Water Waves

For small-amplitude water waves with free surface $z = \eta(x,t)$, the mean curvature may be expanded to leading order as

$$\kappa \;\approx\; -\partial_{xx}\eta,$$

so that the **Laplace–Young** pressure jump becomes

$$P_{\text{fluid}} - P_{\text{air}} = -\gamma\, \partial_{xx}\eta. \tag{8.29}$$

In terms of the perturbation pressure, this modifies our earlier condition at $z = 0$ to

$$\delta P|_{z=0} = -\rho_0 g\, \eta + \gamma\, \partial_{xx}\eta. \tag{8.30}$$

Recalling that $\delta P = -\rho_0 \partial_t \psi$, the dynamic boundary condition becomes

$$\partial_t \psi(x,0,t) = g\, \eta(x,t) - \frac{\gamma}{\rho_0}\, \partial_{xx}\eta(x,t). \tag{8.31}$$

Dispersion Relation with Capillarity

With the same ansatz (11.26) for ψ and η, the boundary conditions yield the modified boundary equation that

$$-i\omega\left(-\frac{i\omega}{k\,\sinh(kH)}\,\eta_0\right)\cosh(kH) = g\,\eta_0 + \frac{\gamma}{\rho_0}k^2\eta_0 \quad\Rightarrow\quad \frac{\omega^2}{k}\,\coth(kH) = g + \frac{\gamma}{\rho_0}k^2. \tag{8.32}$$

$$\boxed{\omega^2 = \left(gk + \frac{\gamma}{\rho_0}\,k^3\right)\,\tanh(kH).} \tag{8.33}$$

- The gk term corresponds to the restoring effect of gravity.
- The $(\gamma/\rho_0)\,k^3$ term corresponds to the restoring effect of surface tension.

Clearly in the **long wavelength extreme**, this reduces the precisely the same behavior that we encountered in the previous section. If instead we go to the short wavelength regime, then we find

$$\omega^2 \sim k^3\tanh(kH) \approx \frac{\gamma H}{\rho_0}k^4.$$

Our takeaway is that $\omega \sim k^2$ means that $v_{\text{phase}} \sim k$ and $v_{\text{group}} \sim 2k$ means that we have a dispersive medium.

R For long waves ($k \to 0$), gravity dominates, and we recover the gravity-wave dispersion relation. For very short waves ($k \to \infty$), the capillary term dominates, and the waves are called *capillary waves*. The crossover occurs at the *capillary length*

$$\ell_c = \sqrt{\frac{\gamma}{\rho_0 g}},$$

where the gravity and surface tension contributions are comparable.

8.5.3 Internal Gravity Waves (Isothermal)

Equilibrium Configuration

Consider an **isothermal atmosphere** with $p(\rho) = K\rho$ under a constant downward acceleration $\mathbf{g} = -g\,\hat{\mathbf{z}}$. *One could work out analogous conditions for an adiabatic (or even more generally polytropic) atmosphere* . Hydrostatic equilibrium implies that

$$0 = -\frac{\nabla p}{\rho} - g\,\hat{\mathbf{z}}.$$

Recognizing that $\nabla p = c_s^2\nabla\rho$, we have

$$c_s^2\,d\log\rho = -g\,dz \implies \rho_0(z) = \rho_S\exp(-z/z_s),$$

where z_s is the **scale height** $z_s = c_s^2/g$. Likewise, the pressure field is related to ρ simply by the equation of state. We have therefore established a fully consistent background fluid solution. We are now ready to introduce perturbation.

Perturbative Analysis

We will solve this particular equation in the **Lagrangian framework** as it is the most expressive of the underlying physics. Consider a set of perturbations $(\Delta\rho, \Delta P, \Delta\mathbf{u})$. In the vertical direction in which the external force is relevant, we have two linearized equations:

$$\begin{aligned} \frac{\partial \Delta\rho}{\partial t} + \frac{\partial(\rho_0 \Delta u)}{\partial z} = \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u}{\partial z} - \frac{\rho_0}{z_s}\Delta u = 0. \\ \frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho_0}\left[c_s^2 \frac{\partial \Delta\rho}{\partial z} + g\Delta\rho\right]. \end{aligned} \tag{8.34}$$

If we take an additional time derivative in the first equation and an additional spatial derivative of the second equation, we find

$$\begin{aligned} \frac{\partial^2 \Delta\rho}{\partial t^2} - \frac{\rho_0}{z_s}\frac{\partial^2 \xi}{\partial t^2} + \rho_0 \frac{\partial^3 \xi}{\partial z \partial t^2} = 0 \\ \rho_0 \frac{\partial^3 \xi}{\partial z \partial t^2} = -\frac{1}{z_s}\left[c_s^2 \frac{\partial \Delta\rho}{\partial z} + g\Delta\rho\right] - \left[c_s^2 \frac{\partial^2 \Delta\rho}{\partial z^2} + g\frac{\partial \Delta\rho}{\partial z}\right] = 0 \end{aligned}$$

Combining these two expressions, we find

$$\frac{\rho_0}{z_s}\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \Delta\rho}{\partial t^2} = -\frac{1}{z_s}\left[c_s^2 \frac{\partial \Delta\rho}{\partial z} + g\Delta\rho\right] - \left[c_s^2 \frac{\partial^2 \Delta\rho}{\partial z^2} + g\frac{\partial \Delta\rho}{\partial z}\right]$$

We now need to simplify. First off,

$$\frac{\rho_0}{z_s}\partial_t^2 \xi = -\frac{1}{z_s}\left[c_s^2 \partial_z \Delta\rho + g\Delta\rho\right],$$

so

$$\boxed{\underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_s^2 \frac{\partial \Delta\rho}{\partial z^2}}_{\text{Classical Wave}} - \underbrace{g\frac{\partial \Delta\rho}{\partial z}}_{\text{Buoyancy}} = 0} \tag{8.35}$$

As such, we see a slightly modified wave appear which has this additional coupling term that seems somewhat interesting. As always, we consider plane wave solutions, which then yield a **dispersion relation** of the form

$$k^2 - \frac{g}{c_s^2}ki - \frac{\omega^2}{c_s^2} = 0 \tag{8.36}$$

Since k dictates the spatial behavior of the wave, we can solve for k in terms of ω using the quadratic formula:

$$\boxed{k = \frac{i}{2z_s} \pm \sqrt{\frac{\omega^2}{c_s^2} - \frac{1}{2z_s^2}}}$$

Now, remember that the **real part of** k determines the **periods of oscillations** while the **imaginary part of** k corresponds to exponentially increasing / decreasing solutions. Thus, we can break this into the imaginary part

$$\kappa = \frac{1}{2z_s},$$

and the real component

$$k = \sqrt{\frac{\omega^2}{c_s^2} - \frac{1}{4z_s^2}}.$$

Our solutions are then of the form

$$\boxed{\Delta\rho(z,t) \;=\; A\, e^{-z/2z_s} \exp[i(kz - \omega t)]} \tag{8.37}$$

where A is a (complex) amplitude set by initial conditions. *Notice that this mimics our expectation in that we see the exponential cutoff behavior that was also present in the structure of the atmosphere itself. Thus, we may propagate waves, but they are always bound in the envelope described by that exponential form.*

Another very interesting feature of this solution comes from the real component of k. We notice that this is **not always a real number**, which means that there are **some scenarios with no oscillatory behavior**!

Definition 8.5.2 — Acoustic cutoff frequency. From the expression for k, we see that oscillatory (propagating) solutions exist only if

$$\omega^2 > \omega_c^2 \equiv \frac{c_s^2}{4z_s^2}.$$

This ω_c is called the **acoustic cutoff frequency**. Frequencies below ω_c correspond to evanescent modes that decay exponentially with height, while frequencies above ω_c propagate as traveling waves.

Intuition

The idea here is that we might create a pressure perturbation on the surface of a planet and create waves. Those waves propagate "infinitely" as plane waves in the directions perpendicular to stratification; however, they are attenuated as they move vertically. This attenuation is **frequency dependent**.

The Eulerian Perturbation Field

To connect with *observable* quantities, we must convert from the Lagrangian to the Eulerian perturbations. Recall the relation

$$\delta\rho = \Delta\rho - \xi_z \frac{\rho_0}{z_s}, \qquad \delta P = c_s^2\, \delta\rho,$$

and that $\Delta u_z = \partial_t \xi_z = -i\omega\xi_z$.

We may compute ξ_z by integrating the continuity equation,

$$\partial_t \Delta\rho + \rho_0\, \partial_z \partial_t \xi_z = 0 \quad \Rightarrow \quad \partial_z \xi_z = \frac{\Delta\rho}{\rho_0}.$$

Integrating with $\Delta\rho \propto e^{-z/2z_s} e^{i(kz-\omega t)}$ gives

$$\xi_z(z,t) = \frac{A}{\rho_S} \frac{e^{z/z_s}}{ik - 1/(2z_s)}\, e^{-z/2z_s}\, e^{i(kz-\omega t)}.$$

Thus, the **Eulerian perturbation fields** are:

$$\boxed{\begin{aligned}&\delta\rho(z,t) = A\, e^{-z/2z_s} e^{i(kz-\omega t)} \left(1 - \frac{1}{z_s\left(ik - \frac{1}{2z_s}\right)}\right),\\ &\delta P(z,t) = c_s^2\, \delta\rho(z,t),\\ &\delta u_z(z,t) = -i\omega\, \xi_z(z,t).\end{aligned}} \tag{8.38}$$

R The Eulerian perturbations share the same oscillatoryexponential structure as the Lagrangian ones, but with an amplitude and phase shift determined by the stratification scale z_s and the wavenumber k. These are the fields one would *observe* at fixed spatial locations in a stratified atmosphere.

8.5.4 Internal Gravity Waves (Adiabatic)

Like the internal gravity waves we discussed above, the same sort of phenomenology can occur in a somewhat richer sense in the context of **adiabatic perturbations**. We will explore this scenario in this section.

The Equilibrium Configuration

We now consider a general fluid in hydrostatic balance under gravity:

$$\frac{\partial P_0}{\partial z} = -g\rho_0.$$

Unlike the isothermal case treated earlier, we do *not* assume any special equation of state for the background. Instead, $P_0(z)$ and $\rho_0(z)$ may arise from any process (e.g. radiative equilibrium, convection, polytropic stratification). The background stratification may therefore differ from what an adiabatic relation alone would prescribe.

The Perturbative Analysis

The crucial assumption is that the *perturbations themselves are adiabatic*, even if the background is not. That is, when a fluid element is displaced, it evolves without exchanging heat with its surroundings. Formally, if the density of a fluid element is perturbed, the pressure change must be adiabatic, so

$$P_0 + \Delta P = P_0 + \left(\frac{\partial P}{\partial \rho}\right)_s \Delta\rho \implies \frac{\Delta P}{P_0} = \left(\frac{d\log P}{d\log\rho}\right)_s \frac{\Delta\rho}{\rho_0}.$$

We **define** the **adiabatic exponent** such that

$$\Gamma_1 = \left(\frac{\partial \log P}{\partial \log \rho}\right)_s.$$

Idea

When we are talking about an adiabatic ideal gas $\Gamma_1 = \gamma$; however, in many stellar scenarios we might have changes in composition or other relevant changes that make Γ_1 only an **effective adiabatic index**.

This condition applies to the *Lagrangian perturbations*. By contrast, the background stratification (P_0, ρ_0) **can have any profile consistent with hydrostatic equilibrium**. The difference between the background stratification and the "adiabatic stratification" encoded by Γ_1 is what gives rise to buoyancy forces and, ultimately, internal gravity waves.

In this case, we're going to solve for the behavior of individual fluid elements in terms of their displacement ξ. From the **continuity equation**,

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial^2 \xi}{\partial t \partial z} = 0.$$

If we integrate in time, we find

$$\frac{\Delta\rho}{\rho_0} = -\frac{\partial \xi}{\partial z}.$$

This will prove to be extremely helpful in our later manipulations.

Let's now also consider the momentum equation. In **Lagrangian form**, this is

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial \Delta P}{\partial z} - g\Delta\rho.$$

We can use the equation of state to connect P and ρ_0. Recall that

$$\Delta P = \Gamma_1 \frac{P_0}{\rho_0} \Delta\rho = \Omega \Delta\rho,$$

so,

$$-\frac{\partial \Delta P}{\partial z} = -\Delta\rho \frac{\partial \Omega}{\partial z} - \Omega \frac{\partial \Delta\rho}{\partial z}.$$

As such, the momentum equation takes the form

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\Delta\rho \frac{\partial \Omega}{\partial z} - \Omega \frac{\partial \Delta\rho}{\partial z} - g\Delta\rho.$$

Substituting in our manipulation of the continuity equation, we find that

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial \xi}{\partial z}\frac{\partial \Omega}{\partial z} + \frac{\Omega}{\rho_0}\left[\rho_0 \frac{\partial^2 \xi}{\partial z^2} + \frac{\partial \xi}{\partial z}\frac{\partial \rho_0}{\partial z}\right] + g\frac{\partial \xi}{\partial z}$$

We can now group like terms to obtain the wave equation in ξ:

$$\frac{\partial^2 \xi}{\partial t^2} - \Omega \frac{\partial^2 \xi}{\partial z^2} = \frac{\partial \xi}{\partial z}\left[\frac{\partial \Omega}{\partial z} + \Omega \frac{\partial \log \rho_0}{\partial z} + g\right] \tag{8.39}$$

This is incomplete

8.6 Boundary Conditions

Consider two uniform media sharing a boundary. Let each of them be governed by a **barotropic equation of state** so that $p = K_1\rho$ and $p = K_2\rho$ in each of the media respectively. In order for the behavior at the interface to be sensible, we require that all of the relevant variables (ρ, p, u) be **continuous** and single valued. Consider a general solution to the standard wave equation in the first medium:

$$\Delta\rho = I\exp(i[kx - \omega t]) + R\exp(i[kx + \omega t]).$$

Likewise, consider a wave in the second media

$$\Delta\rho_2 = T\exp(i[k_2 x - \omega_2 t]).$$

If the interface is at $x = 0$, then for continuity,

$$Ie^{-i\omega t} + Re^{i\omega t} = Te^{i\omega_2 t}.$$

We clearly see that the waves must have **the same angular frequency**. Thus,

$$Ie^{-i\omega t} + Re^{i\omega t} = Te^{i\omega t} \implies I + R = T.$$

In a uniform media, we also need continuity of the derivative, so

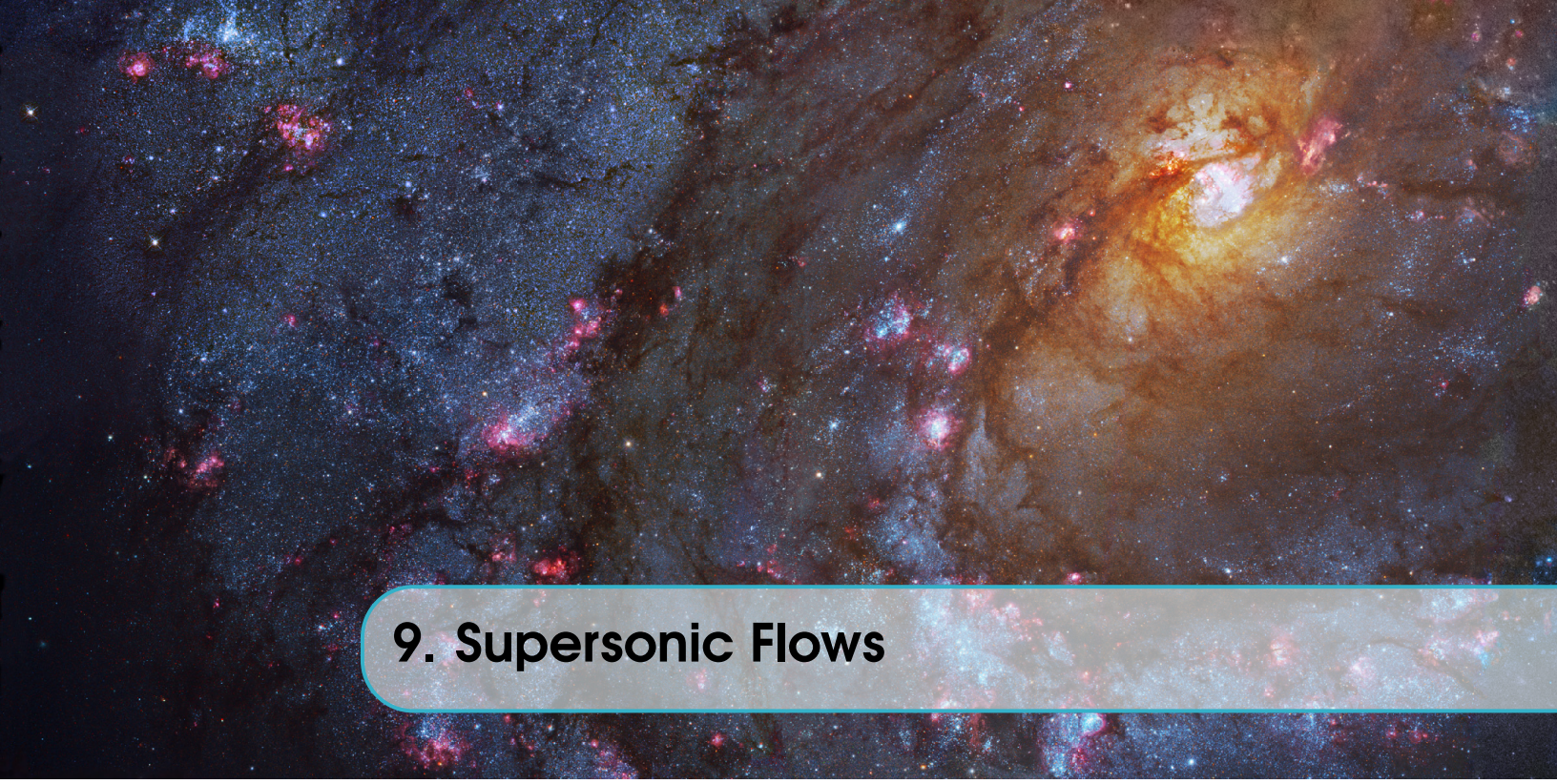
$$Iik_1 - Rik_1 = iTk_2 \implies (I - R)k_1 = Tk_2.$$

Letting $I = 1$, we have

$$1 + R = T, \text{and } (1 - R)k_1 = k_2 T,$$

so

$$T = \frac{2k_1}{k_1 + k_2},\ R = \frac{k_1 - k_2}{k_1 + k_2}.$$



9. Supersonic Flows

9.1 Intuition

Supersonic flows occur when a **disturbance propagates faster than the local speed of sound**. In this regime, information cannot propagate upstream to "warn" the undisturbed medium, and discontinuities in the flow (shock waves) form.

Consider the following simple scenario: you (the observer) sit at the origin $(0, 0)$, while a uniform flow with velocity v and sound speed c_s moves past you in the $+x$ direction. At some time $t_0 = 0$, you insert a tuning fork into the flow, generating periodic disturbances with period τ.

R Intuitively, if the disturbance speed is much smaller than the sound speed ($v \ll c_s$), then the tuning fork generates small-amplitude density and pressure waves that propagate outward through the fluid at speed c_s in all directions (in the fluid frame). Because these waves travel much faster than the source itself, they can reach the upstream fluid before the source arrives, giving the medium time to adjust smoothly. As a result, successive wavefronts remain separated, and the flow changes continuously without sharp discontinuities.

In contrast, if the disturbance speed is much greater than the sound speed ($v \gg c_s$), the fluid upstream cannot receive any "advance notice" of the approaching source. The chain of particle collisions that transmits pressure changes cannot outpace the source, so the medium only reacts when the source is already upon it. This leads to a pile-up of wavefronts into a narrow region where the density, pressure, and velocity change abruptly — a shock front. In the supersonic regime, the envelope of these piled-up wavefronts forms the Mach cone downstream of the disturbance.

In the frame of the fluid, each disturbance propagates spherically with speed c_s. The center of the k-th disturbance in the lab frame is located at:

$$x_{\text{center}} = k\,\tau\,v.$$

Thus, the equation for the k-th wavefront is

$$(x - k\tau v)^2 + y^2 = c_s^2\,(t - k\tau)^2. \tag{9.1}$$

At the downstream edge of the k-th wavefront, we have:

$$x - k\tau v = c_s\,(t - k\tau), \tag{9.2}$$

$$\implies x_k = c_s t + k\tau(v - c_s). \tag{9.3}$$

The longitudinal spacing between successive wavefronts is therefore:

$$\Delta x = x_{k+1} - x_k = \tau(v - c_s). \tag{9.4}$$

We can now classify the flow regimes:

1. **Subsonic** ($v < c_s$): Wavefronts remain in order; disturbances can propagate upstream.
2. **Sonic** ($v = c_s$): Wavefronts "pile up" downstream, producing a stationary compression region.
3. **Supersonic** ($v > c_s$): Successive wavefronts overtake one another downstream, forming a shock front.

9.1.1 Mach Cone Formation

In the supersonic case ($v > c_s$), the envelope of all wavefronts forms a conical shock surface,the *Mach cone* — that trails the disturbance in the downstream direction.

At time t, the furthest extent of any disturbance in the y-direction is

$$y_{\max} = c_s t,$$

while the furthest downstream extent in x is

$$x_{\max} = vt.$$

The half-opening angle θ of the Mach cone is therefore:

$$\sin\theta = \frac{c_s}{v}. \tag{9.5}$$

Definition 9.1.1 — Mach Number. The *Mach number* M is the ratio of the object's speed to the local speed of sound:

$$M \equiv \frac{v}{c_s}.$$

In terms of M, the Mach angle is:

$$\theta = \sin^{-1}\left(\frac{1}{M}\right).$$

A larger Mach number corresponds to a narrower cone, while $M \to 1^+$ corresponds to a very wide, weak cone.

9.2 The Rankine-Hugoniot Conditions

Consider a **shock front** dividing two regions of fluid. We refer to each side as the **upstream** and **downstream** side of the shock front. For the sake of simplicity, we assume the front occurs at $x = 0$ in some reference frame and that, on either side of the front, we have some $\rho_{1,2}, p_{1,2}$, and $u_{1,2}$. Now, certain conservation laws must still be true, namely those which provide us with the **Euler Equations**. As such, we can define some elements of the behavior across the shock front in terms of these conservation rules.

R An **critical** realization is that the Rankine-Hugoniot relations which we are soon to derive are valid **only in the rest frame of the shock**. As such, we are always implicitly performing a Galilean transformation into that frame when we use them.

In many cases, this makes the intuition for which side of the shock is which tricky: if a bow shock is driven into the ICM of a galaxy cluster, the gas in the galaxy cluster is the **upstream side** since it moves towards the shock in the shock's reference frame.

The Continuity Condition

In Eulerian form the continuity equation is,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$

if we integrate across some infinitesmal width δx on either side of the shock front,

$$\frac{\partial}{\partial t} \int_{-\delta x}^{\delta x} \rho \, dx + (\rho u)_{\delta x} - (\rho u)_{-\delta x} = 0.$$

Now, as $\delta x \to 0$, we clearly have that the integral term vanishes and

$$\boxed{\rho_1 u_1 = \rho_2 u_2}.$$

The Momentum Condition

In one-dimensional inviscid flow with an external body force $\mathbf{f}_{\text{ext}}$, the momentum equation in *conservative form* is

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = \rho \mathbf{f}_{\text{ext}}. \tag{9.6}$$

Here ρu is the momentum density, and $\rho u^2 + p$ is the momentum flux (mass flux of momentum plus the pressure force).

We now integrate this equation across a thin control volume enclosing a discontinuity at $x = 0$, extending from $x = -\delta x$ to $x = +\delta x$. In the shock rest frame (steady state), the time derivative vanishes upon integration:

$$\int_{-\delta x}^{+\delta x} \frac{\partial}{\partial x}\left(\rho u^2 + p\right) dx = \int_{-\delta x}^{+\delta x} \rho \mathbf{f}_{\text{ext}} \, dx. \tag{9.7}$$

If $\mathbf{f}_{\text{ext}}$ is bounded, its contribution is $O(\delta x)$ and vanishes as $\delta x \to 0$. Therefore, in the limit we obtain

$$\left[\rho u^2 + p\right]_1^2 = 0, \tag{9.8}$$

where $[A]_1^2 \equiv A_2 - A_1$ denotes the jump across the discontinuity. This is the **momentum RankineHugoniot condition**:

$$\boxed{\rho_1 u_1^2 + p_1 \;=\; \rho_2 u_2^2 + p_2.} \tag{9.9}$$

An equivalent statement may be created where the force is not bounded. See, for example, Thorne+Blandford.

The Energy Condition

To derive the relevant condition for energy conservation across a shock, we adopt two simplifying assumptions:

1. **Adiabatic flow**: there is no external heating or cooling, so we may ignore source terms in the energy equation.

2. **Inviscid flow**: viscous dissipation is neglected.

Under these assumptions, the energy equation takes the form

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\,\mathbf{u}] = 0,$$

where $E = \frac{1}{2}\rho u^2 + \rho\epsilon$ is the total energy density, with ϵ the specific internal energy.

This equation can also be expressed in terms of enthalpy, $h \equiv \epsilon + p/\rho$, as

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\left(\tfrac{1}{2}\rho u^2 + \rho h\right)\mathbf{u}\right] = 0.$$

Assuming steady state and integrating across a vanishingly thin control volume that encloses the shock, we obtain equivalent jump conditions:

$$\begin{aligned} \left[u(E+p)\right]_1^2 &= 0 \qquad \text{(flux of total energy + pressure)} \\ \left[u(\tfrac{1}{2}\rho u^2 + \rho h)\right]_1^2 &= 0 \qquad \text{(flux of kinetic + enthalpy energy).} \end{aligned} \tag{9.10}$$

Finally, using the mass conservation condition $\rho u = \text{const}$ across the shock, we can eliminate u and write the energy condition in more compact forms:

$$\boxed{\left[\mathcal{E} + \tfrac{p}{\rho}\right]_1^2 = 0}, \qquad \text{or equivalently} \qquad \boxed{\left[\tfrac{1}{2}u^2 + h\right]_1^2 = 0}. \tag{9.11}$$

Here $\mathcal{E} = \epsilon + \frac{1}{2}u^2$ is the specific total energy.

9.2.1 The Rankine–Hugoniot Conditions

Definition 9.2.1 — Rankine–Hugoniot Conditions. The *Rankine–Hugoniot (RH) conditions* express the conservation of mass, momentum, and energy across a steady, planar shock front. They are valid only in the **rest frame of the shock**, where the discontinuity is stationary and the upstream fluid flows into the front. Together, they constrain the allowed discontinuities in density, velocity, pressure, and temperature.

For upstream quantities (ρ_1, u_1, p_1) and downstream quantities (ρ_2, u_2, p_2), the RH conditions are:

1. **Mass conservation (continuity):**

$$\rho_1 u_1 = \rho_2 u_2 \tag{9.12}$$

The mass flux through the shock is the same on both sides.

2. **Momentum conservation:**

$$\rho_1 u_1^2 + p_1 \;=\; \rho_2 u_2^2 + p_2. \tag{9.13}$$

The sum of momentum flux and pressure force is conserved.

3. **Energy conservation:**

$$\frac{1}{2}u_1^2 + h_1 \;=\; \frac{1}{2}u_2^2 + h_2, \tag{9.14}$$

where $h = \epsilon + p/\rho$ is the specific enthalpy. This states that the specific total energy (kinetic plus thermal) is continuous across the shock.

Together, these three relations define the *Rankine–Hugoniot conditions*. They show that shocks are not arbitrary discontinuities: only those jumps that satisfy mass, momentum, and energy conservation are physically possible.

9.2.2 Additional Forms of the RH Conditions

Now, in their current form, equations 9.14, (9.13), and (9.14) are dependent on the flow velocity, the internal energy, and the density / pressure. In many scenarios, **these are not all measurable properties of the flow** and instead we seek to find a simpler / more useful way to cast these relationships. The first step in doing so is to better understand the internal energy E which appears in the above equations. Let us formally **assume** a polytropic equation of state of the form,

$$p = \rho^{\Gamma},$$

The convenience of this assumption is that the enthalpy is

$$h = \int^{P} \frac{dP}{\rho} = \int^{\rho} \Gamma \rho^{\Gamma-2}\, d\rho = \frac{\Gamma}{\Gamma - 1}\rho^{\Gamma-1}.$$

Noting that

$$c_s^2 = \frac{\partial P}{\partial \rho} = \Gamma \rho^{\Gamma-1} \implies \boxed{h = \frac{c_s^2}{\Gamma - 1}.}$$

We can therefore get the very convenient form of the energy condition (9.14):

$$\boxed{\frac{1}{2}u_1^2 + \frac{c_1^2}{\Gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_2^2}{\Gamma - 1}} \tag{9.15}$$

This form of the **Rankine-Huginiot condition** is already quite nice, but there are many manipulations to be made to these expressions in order to get various forms worth exploration. At this stage, it is worth developing something of a heuristic picture of how these can be used.

We have, in this form of the RH conditions, 4 sets of variables: $\rho_{[1,2]}$, $u_{[1,2]}$, $c^2_{[1,2]}$, and $P_{[1,2]}$. Now, the RH conditions (and the EOS) provide the following relationships:

1. **Continuity**: Relates ρ and u.
2. **Momentum**: Relates P and u.
3. **Energy**: Relates u and c_s^2.
4. **EOS**: Relates P and ρ.

Mach Number Form of the RH Conditions

It is often useful to reexpress the Rankine–Hugoniot conditions in terms of the **Mach number**,

$$M \equiv \frac{u}{c_s},$$

which combines the flow speed u and the sound speed c_s into a single dimensionless variable. This is particularly helpful in astrophysical contexts, where shocks are commonly characterized by their upstream Mach number M_1.

Starting from the energy condition across the shock, we can write

$$\frac{1}{2} + \frac{M_1^{-2}}{\Gamma - 1} = \frac{1}{2}\frac{u_2^2}{u_1^2} + \frac{(c_2^2/u_1^2)}{\Gamma - 1},$$

where Γ is the adiabatic index.

From the continuity condition ((9.12)) we know that

$$\frac{\rho_1}{\rho_2} = \frac{u_2}{u_1} \equiv x,$$

so that

$$\frac{1}{2} + \frac{M_1^{-2}}{\Gamma - 1} = \frac{1}{2}x^2 + \frac{c_2^2}{c_1^2}\frac{M_1^{-2}}{\Gamma - 1}.$$

Rearranging gives

$$\frac{1}{2}(1 - x^2) = \frac{M_1^{-2}}{\Gamma - 1}\left(\frac{c_2^2}{c_1^2} - 1\right).$$

The ratio of sound speeds follows from the equation of state:

$$\frac{c_2^2}{c_1^2} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{P_2}{P_1}x.$$

To eliminate P_2/P_1, we use the momentum Rankine–Hugoniot condition:

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2.$$

Dividing through by P_1 and using $x \equiv u_2/u_1 = \rho_1/\rho_2$, we can rewrite the right-hand side:

$$\frac{\rho_2 u_2^2}{P_1} = \frac{\rho_2 (x u_1)^2}{P_1} = \frac{\rho_1 u_1^2}{P_1}\, x.$$

Therefore the momentum condition becomes

$$\frac{\rho_1 u_1^2}{P_1} + 1 = \frac{\rho_1 u_1^2}{P_1}\, x + \frac{P_2}{P_1}.$$

Rearranging gives

$$1 - \frac{P_2}{P_1} = \frac{\rho_1 u_1^2}{P_1}(x - 1).$$

Next, we express the prefactor in terms of the Mach number. Since

$$M_1^2 = \frac{u_1^2}{c_1^2}, \qquad c_1^2 = \frac{\Gamma P_1}{\rho_1},$$

we have

$$\frac{\rho_1 u_1^2}{P_1} = \frac{\rho_1}{P_1} M_1^2 c_1^2 = \frac{\rho_1}{P_1} M_1^2 \frac{\Gamma P_1}{\rho_1} = \Gamma M_1^2.$$

Thus the pressure jump condition is

$$\frac{P_2}{P_1} = 1 + \Gamma M_1^2 (1 - x).$$

Finally, recalling that $c^2 = \Gamma P/\rho$, the sound speed ratio may be expressed as

$$\frac{c_2^2}{c_1^2} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{P_2}{P_1}\, x = x\left[1 + \Gamma M_1^2 (1 - x)\right].$$

Substituting this into the modified energy equation yields

$$\frac{M_1^2}{2}(x^2 - 1) = \frac{1}{\Gamma - 1}\left[\Gamma x (1 - x) M_1^2 + x - 1\right].$$

Factoring out $(x - 1)$ from the modified energy equation and simplifying gives

$$\frac{1}{2}(\Gamma - 1) M_1^2 (x + 1) + \Gamma x - M_1^2 = 0.$$

Solving this quadratic relation for $x = \rho_1/\rho_2$ and inverting, we obtain the classic **compression ratio** across a shock:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\Gamma + 1) M_1^2}{(\Gamma - 1) M_1^2 + 2}} \tag{9.16}$$

This compact expression shows that the density jump depends only on the upstream Mach number. In the strong shock limit ($M_1 \to \infty$), the ratio saturates at $\rho_2/\rho_1 = (\Gamma+1)/(\Gamma-1)$, which equals $\boxed{4 \text{ for a monatomic ideal gas } (\Gamma = 5/3)}$. As the shock weakens, the density ratio becomes 1.

From the derivation above, we can also write down the **pressure ratio** in the form

$$\boxed{\frac{P_2}{P_1} = 1 + \frac{2\Gamma}{\Gamma + 1}\left(M_1^2 - 1\right)}, \tag{9.17}$$

where we have used the fact that $x = u_2/u_1 = \rho_1/\rho_2$ and the expression we derived from P_2/P_1 in terms of x above.

The Density-Pressure Form

It is sometimes convenient to eliminate the Mach number entirely and express the compression ratio ρ_2/ρ_1 directly in terms of the pressure ratio P_2/P_1. Starting from the momentum condition in dimensionless form,

$$\frac{P_2}{P_1} = 1 + \Gamma M_1^2 (1 - x),$$

with $x \equiv \rho_1/\rho_2$, we can rearrange this relation to isolate M_1^2:

$$M_1^2 = \frac{\frac{P_2}{P_1} - 1}{\Gamma(1-x)}.$$

On the other hand, from the density ratio expressed in terms of Mach number (eq. 9.16),

$$\frac{\rho_2}{\rho_1} = \frac{(\Gamma+1)M_1^2}{(\Gamma-1)M_1^2+2}.$$

Substituting the above expression for M_1^2 into this equation and simplifying yields a direct relation between the density and pressure ratios:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{P_2}{P_1} + \frac{\Gamma-1}{\Gamma+1}}{\frac{\Gamma}{\Gamma+1}\frac{P_2}{P_1} + \frac{1}{\Gamma+1}}.$$

Equivalently, this can be written in a slightly cleaner form:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\Gamma-1)P_1 + (\Gamma+1)P_2}{(\Gamma+1)P_1 + (\Gamma-1)P_2}} \tag{9.18}$$

This form of the Rankine–Hugoniot condition is particularly useful in practice: if the pressure jump across a shock is measured (for example in X-ray observations of galaxy clusters), the corresponding compression ratio of the gas can be inferred directly.

The Temperature RH Conditions

The behavior of the temperature across the RH conditions is determined from the other ratios we have already derived. Specifically,

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \cdot \frac{\rho_1}{\rho_2}.$$

Substituting the expressions for the pressure and density ratios in Mach form,

$$\frac{P_2}{P_1} = 1 + \frac{2\Gamma}{\Gamma+1}\left(M_1^2 - 1\right), \qquad \frac{\rho_2}{\rho_1} = \frac{(\Gamma+1)M_1^2}{(\Gamma-1)M_1^2+2},$$

we arrive at

$$\frac{T_2}{T_1} = \left[1 + \frac{2\Gamma}{\Gamma+1}\left(M_1^2 - 1\right)\right]\left[\frac{(\Gamma-1)M_1^2+2}{(\Gamma+1)M_1^2}\right].$$

Simplifying gives the compact form

$$\boxed{\frac{T_2}{T_1} = \frac{\left[2\Gamma M_1^2 - (\Gamma-1)\right]\left[(\Gamma-1)M_1^2+2\right]}{(\Gamma+1)^2 M_1^2}} \tag{9.19}$$

As a check, in the strong-shock limit $M_1 \to \infty$,

$$\frac{T_2}{T_1} \longrightarrow \frac{2\Gamma(\Gamma-1)}{(\Gamma+1)^2} M_1^2.$$

For a monatomic ideal gas ($\Gamma = 5/3$), this reduces to

$$\frac{T_2}{T_1} \longrightarrow \frac{5}{16} M_1^2.$$

9.3 Isothermal Shocks

In the adiabatic treatment above, we neglected cooling in the energy equation. This is not always valid. When the gas is strongly coupled to a cooling process (e.g. radiation, conduction to a cold reservoir), thermal energy generated in the shock is **removed rapidly and the temperature remains approximately constant across the discontinuity**. Such shocks are well described by the *isothermal* limit.

R Adiabatic shocks apply when the cooling time is long compared to the advection time through the shock layer; isothermal shocks apply in the opposite limit:

$$t_{\rm cool} \gg t_{\rm adv} \quad \text{(adiabatic)}, \qquad t_{\rm cool} \ll t_{\rm adv} \quad \text{(isothermal)}.$$

The conservative forms of the Euler equations (1D, steady, shock at $x = 0$) are

$$\text{Mass:} \qquad \frac{d}{dx}(\rho u) = 0 \implies \rho_1 u_1 = \rho_2 u_2 \equiv m', \tag{9.20}$$

$$\text{Momentum:} \qquad \frac{d}{dx}\left(\rho u^2 + p\right) = 0 \implies \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2, \tag{9.21}$$

$$\text{Energy (with cooling):} \qquad \frac{d}{dx}\left[u\,(E+p)\right] = -\,\mathcal{L}(x), \tag{9.22}$$

where $E = \rho e + \frac{1}{2}\rho u^2$ is the total energy density and $\mathcal{L}$ is the (positive) volumetric cooling rate.

Integrating (9.22) across the thin shock layer gives the *corrected* energy jump:

$$\left[u(E+p)\right]_1^2 \;=\; -\int_{-\delta x}^{+\delta x} \mathcal{L}(x)\,dx. \tag{9.23}$$

In the **isothermal limit**, cooling is efficient enough to maintain $T_2 \approx T_1 \equiv T$, so the ideal-gas equation of state reads

$$p = \rho c_s^2, \qquad c_s^2 \equiv \frac{k_B T}{\mu} = \text{const.} \tag{9.24}$$

In practice, one then uses (9.20), (9.21), and (9.24); the energy jump (9.23) is *implicitly* satisfied by the cooling that enforces $T = \text{const}$.

Insert $p = c_s^2 \rho$ into the momentum jump (9.21) and use $m' = \rho u$:

$$\rho_1 u_1^2 + c_s^2 \rho_1 = \rho_2 u_2^2 + c_s^2 \rho_2.$$

With $u_i = m'/\rho_i$ this becomes

$$\frac{m'^2}{\rho_1} + c_s^2 \rho_1 = \frac{m'^2}{\rho_2} + c_s^2 \rho_2 \implies m'^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = c_s^2\,(\rho_2 - \rho_1).$$

For a nontrivial jump ($\rho_2 \neq \rho_1$), cancel $(\rho_2 - \rho_1)$ to obtain

$$m'^2 = c_s^2\,\rho_1 \rho_2. \tag{9.25}$$

Using $m' = \rho_1 u_1$ or $m' = \rho_2 u_2$ then gives the classic isothermal relations:

$$\frac{\rho_2}{\rho_1} = \frac{u_1^2}{c_s^2} = M_1^2, \tag{9.26}$$

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{1}{M_1^2}, \tag{9.27}$$

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} = M_1^2, \tag{9.28}$$

where $M_1 \equiv u_1/c_s$ is the *isothermal Mach number* upstream. Thus, for an isothermal shock the compression (and pressure) ratio is simply M_1^2.

R In contrast to adiabatic shocks (where the maximum compression is finite, e.g. $\rho_2/\rho_1 \leq (\gamma+1)/(\gamma-1)$ for $\gamma > 1$), an isothermal shock can in principle achieve arbitrarily large compression as M_1 increases. The trade-off is that the shock must radiate away the corresponding thermal energy to keep T fixed.

Cooling Length and Relevance

Let $\mathcal{L}$ be the volumetric cooling rate and define an *isobaric* or *isochoric* cooling time $t_{\rm cool}$ appropriate to the downstream state (model-dependent). The *cooling length* is the distance over which the post-shock flow loses the shock-generated thermal energy:

$$\ell_{\rm cool} \sim u_2\, t_{\rm cool}. \tag{9.29}$$

The isothermal approximation is valid if the thermal energy is removed on a length scale short compared to the advection/dynamical scale of interest L:

$$\ell_{\rm cool} \ll L \quad \Longleftrightarrow \quad t_{\rm cool} \ll \frac{L}{u_2}.$$

Physically: microscopic collisions in the shock layer still convert bulk kinetic energy into random motion, but efficient cooling immediately removes that energy, preventing a temperature rise and enforcing the equation of state $p = c_s^2 \rho$.

10. Blastwaves

A common scenario in both theory and application is that of the **blast-wave** created by depositing energy E into a medium effectively instantaneously and seeing the evolution. When E is large, the inevitable result is a **blast-wave** led by a **shock front**. It is the properties of such shocks that we will study in this chapter. To begin, consider the following scenario:

Consider an equilibrated region of fluid subject to no external gradients and with temperature $T \sim 0$. When a large quantity E_0 of energy is deposited at the center of the medium, the result is a shock wave which sweeps up the material.

10.1 The Blastwave Scenario

- set up the system, define the rest and shock frame, the u's, p's, and rho's - the energy injection - the basic idea of how we want to proceed.

criteria

- We need to give time for the ejecta to settle and we don't have the ejecta continuing. This is the Sedov period of a SN phase. - Must be energy conserving phase, where radiative processes become relevant. Afterward that is late phase. this is often 100s - 1000s of years in SN explosions.

10.2 The Approximate Method

As the explosion propagates, we have a shock wave which will be a **strong shock** (*the argument is that enough energy is injected which makes the post-shock region so much hotter, so the velocity ratio must be very large*). To be clear about our notation and conventions, we identify two regions:

- The **downstream region** is **inside the shock**: this is the material which makes up the shell of the ejecta.
- The **upstream region** is **outside the shock**: this is the ambient material which is (in the frame of the shock) going to flow into it.

Because the shock is **strong**, we adopt the treatment that $M = \infty$ and, therefore, the **Rankine-Huginiot conditions** require that (equation (9.16))

$$\frac{\rho_1}{\rho_0} = \frac{\gamma+1}{\gamma-1},$$

where ρ_0 is the **upstream (ambient) density** and ρ_1 is the **downstream (shock front) density**. This is useful because it permits us to calculate the width of the shock: assuming that all of the mass is carried outward by the explosion, it will displace $(4/3)\pi\rho_0 R^3$ (*since ρ_0 was the ambient density everywhere*) mass, which then must be contained in a shell of thickness D and density ρ_1, so

$$4\pi D R^2 \rho_1 = \frac{4\pi R^3 \rho_0}{3} \implies D = \frac{R}{3}\frac{\rho_0}{\rho_1} = \frac{R}{3}\frac{\gamma-1}{\gamma+1}.$$

For a reasonable gas, $\gamma \sim 1$ and so if $\gamma = 1+\chi$, then

$$\frac{\gamma-1}{\gamma+1} = \frac{\chi}{2+\chi} \ll 1.$$

As such, we conclude that the shock width D is **quite thin**. In making that assumption, we can also assume that the entire shell moves with the same velocity. Thus, **in the frame of the shock**, the RH conditions require that

$$\rho_0 u_0 = u_1 \rho_1 \implies u_1 = \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma - 1}{\gamma + 1} u_0.$$

Now, relative to the ambient environment (velocity u_1), the shock has velocity $U = u_0 - u_1$, so

$$U = u_0 - u_1 = \frac{2u_0}{\gamma + 1}.$$

This is then telling us how the actual rest-frame shock velocity behaves. The **momentum of the shock** is then

$$p_{\text{shock}} = \underbrace{4\pi D R^2}_{\text{volume}} \underbrace{\rho_1 U}_{\text{momentum density}} = \frac{4}{3}\pi R^3 \rho_0 \frac{2u_0}{\gamma + 1}.$$

Clearly,

$$\dot{p}_{\text{shock}} = \frac{8\pi\rho_0}{3(\gamma+1)} \frac{d}{dt}(u_0 R^3) \geq 0.$$

So the shock is **gaining momentum**. *Remember that, in the rest frame of the shock, $u_0 \gg 0$ and R is increasing* . This must be because there is a pressure gradient at play! Now, we need the pressure P_{in} to drive the shock outward, which means it must be higher than P_1 (the post-shock pressure), and obviously higher than P_0 (which is zero since we set the temperature to 0). We consider the ansatz

$$P_{\text{in}} = \alpha P_1.$$

This is not an entirely justifiable assumption, but it serves to illustrate things. For an adiabatic shock, the RH conditions give us that

$$\rho_0 u_0^2 + p_0 = \rho_1 u_1^2 + p_1.$$

We assume that $p_0 = 0$, so

$$p_1 = \rho_0 u_0^2 - \rho_1 u_1^2 = \rho_0 u_0^2 - \frac{\gamma - 1}{\gamma + 1}\rho_0 u_0^2 = \frac{2}{\gamma + 1}\rho_0 u_0^2.$$

Thus, if $P_{\text{in}} = \alpha p_1$, then

$$\frac{8\pi\alpha}{\gamma + 1} R^2 \rho_0 u_0^2 = \dot{p}_{\text{shock}} = \frac{8\pi\rho_0}{3(\gamma+1)} \frac{d}{dt}\left(u_0 R^3\right),$$

so

$$3\alpha R^2 u_0^2 = \frac{d}{dt}\left(u_0 R^3\right).$$

Now,

$$u_0 = \frac{dR}{dt},$$

since *u_0 is the ambient gas velocity in the **frame of the shock**, which means that it is moving into the shock at the same speed the shock is moving through the medium.* Thus,

$$3\alpha R^2 \dot{R}^2 = \frac{d}{dt} R^3 \dot{R} = 3R^2 \dot{R}^2 + R^3 \ddot{R}.$$

Let's assume a **power law solution:**

$$R \propto t^\beta.$$

Then,

$$3\alpha\beta^2 t^{4\beta-2} = 3\beta^2 t^{4\beta-2} + \beta(\beta-1)t^{4\beta-2}$$

so

$$(3\alpha - 4)\beta = -1 \implies \beta = \frac{1}{4-3\alpha}.$$

So

$$\boxed{R \sim t^{1/(4-3\alpha)}}$$

and

$$\boxed{u_0 \sim t^{(3\alpha-3)/(4-3\alpha)} \sim R^{3\alpha-3}.}$$

Now, we come to the next clear question: **what is α?** To determine this, we rely on the energetics of the explosion. If the blast wave is **adiabatic**, then it loses no energy as it evolves, so we need that E_0 to convert into internal energy and kinetic energy. Clearly, the kinetic energy of the wave is

$$K = \frac{1}{2}\frac{4\pi}{3}\rho_0 R^3 U^2 = \frac{8\pi}{3(\gamma+1)^2}\rho_0 u_0^2 R^3.$$

Likewise, the internal energy needs to be computed. Now, for an adiabatic gas, we likewise have

$$p = K\rho^\gamma,$$

and the first law of thermodynamics implies that

$$dU = -pdV \implies d\mathcal{E} = -K\rho^\gamma d\log V,$$

but $V = m/\rho \implies dV = m d\log\rho = V d\log\rho$, so

$$d\log\rho = -d\log V,$$

and

$$d\mathcal{E} = K\rho^{\gamma} d\log\rho = K\rho^{\gamma-1}d\rho,$$

so

$$\mathcal{E} = p/(\gamma-1).$$

Now, we assumed (because the shock is thin) that the bulk of the internal energy is located in the less dense post-shocked gas, which means that we have internal energy

$$U = \frac{4}{3}\pi R^3 \frac{P_{\text{in}}}{\gamma-1}.$$

If we now write the entire energy in terms of time t, we have

$$E = \frac{4}{3(\gamma-1)}\pi R^3\left[\alpha p_1 + \frac{2}{\gamma+1}\rho_0 u_0^2\right]$$
$$E = \frac{4}{3(\gamma-1)(\gamma+1)}\pi R^3\left(\alpha+2\right)\rho_0 u_0^2,$$

so

$$E \sim R^3 u_0^2 \sim t^{(6\alpha-3)/(4-3\alpha)},$$

which, for **energy conservation**, requires that $\alpha = 0.5$, so

$$\boxed{R \sim t^{2/5},\ u_0 \sim t^{-3/5},\ p_1 \sim t^{-6/5}.} \tag{10.1}$$

In the next section, we will explore a fully rigorous solution.

10.3 Similarity Solution

Let us now consider a rigorous treatment of the material within the blast. We consider $v(r,t)$, $\rho(r,t)$, and $p(r,t)$. The **Euler Equations** take the form

$$\partial_t v + v\partial_r v = -\partial_r p/\rho$$
$$\partial_t \rho + \partial_r(\rho v) = -2\rho v/r$$
$$(\partial_t + v\partial_r)\log p/\rho^{\gamma} = 0$$

they look this way because they are in spherical symmetry. We are assuming an adiabatic system. Now, it can be observed that for the blast wave problem we have been considering, these equations are **scale invariant**, meaning that there is no natural length scale to the problem. We therefore cannot permit a solution to insert some arbitrary scale into the problem where none was before. As such, we will cast the problem in terms if the **only available dimensionless scale**:

$$\xi = r\left(\frac{\rho_0}{E_0 t^2}\right)^{1/5}.$$

Now, we will endeavor to find solutions to our equations which place the shock front at $\xi = 1$ and the center of the blast at $\xi = 0$. Clearly, if R is the blast radius, then

$$R(t) = \left(\frac{E_0 t^2}{\rho_0}\right)^{1/5}.$$

The velocity of the shock wave relative to the lab frame is then

$$\dot{R} = \frac{2}{5}\frac{R}{t}.$$

Now, we would like to understand the various **post-shock quantities** (index 1) in terms of the **pre-shock quantities** (index 0) and our derived attributes. These are provided by the **Rankine-Huginiot Conditions**, which provide us with connections between the relevant quantities. We have (in the case of a strong shock)

$$\begin{aligned} u_1 &= \frac{2}{\gamma+1}\dot{R} \\ p_1 &= \frac{2}{\gamma+1}\rho_0 \dot{R}^2 \\ \rho_1 &= \rho_0 \frac{\gamma+1}{\gamma-1} \end{aligned}$$

Let's now establish a game plan: behind the shock, each quantity $\psi(r,t)$ is going to scale from ψ_1 right behind the shock at $\xi = 1$, so what if we define the quantity as

$$\psi(r,t) = \psi(\xi) = \psi_1 \cdot \tilde{\psi}(\xi),$$

where $\tilde{\psi}$ is the **scaling function** for that particular quantity, which is then determined from the differential equation. It turns out that you can in fact solve for these scaling terms. They then need to be integrated properly to get the solutions.

Add more detail here.

Now, if we assume that each of our various quantities is scale invariant, then we can assume equations of the form

$$\begin{aligned} \xi &= r/R \\ V(\xi) &= \frac{5}{2}\frac{t}{r}u_1 \\ G(\xi) &= \frac{\rho}{\rho_0} \\ Z(\xi) &= \frac{25 t^2 c_s^2}{4r^2}. \end{aligned}$$

the constant factors just make things turn out nicely. The relevant parts here are that we combine r, t, ρ_0, c_s to facilitate the correct dimensions. With $\xi = 1$ acting as the boundary, we can derive a set of ODEs. These equations can be numerically integrated when relevant.

10.4 Breakdown of the Blast Wave

At a certain point, the blast wave solution will diminish because the condition that there be negligible pressure outside the blast will no longer be satisfied. Formally, once the blast wave is moving at $\sim c_s$ in the ambient medium, the shock will fail to hold and the wave behavior will take over, propogating the disturbance outward.

Formally, if we set the criterion that $p_1 \sim p_0 = \rho_0 c_s^2/\gamma$, then $u_0^2 \sim c_s^2(\gamma+1)/(2\gamma)$. Using conservation of energy, we can compute the radius at which this happens as

$$E = \frac{4\pi}{3} R^3 \left[\frac{1}{2}\rho \left(\frac{2u_0}{\gamma+1} \right)^2 + \frac{\alpha}{\gamma-1} \frac{2\rho_0 u_0^2}{\gamma+1} \right]$$

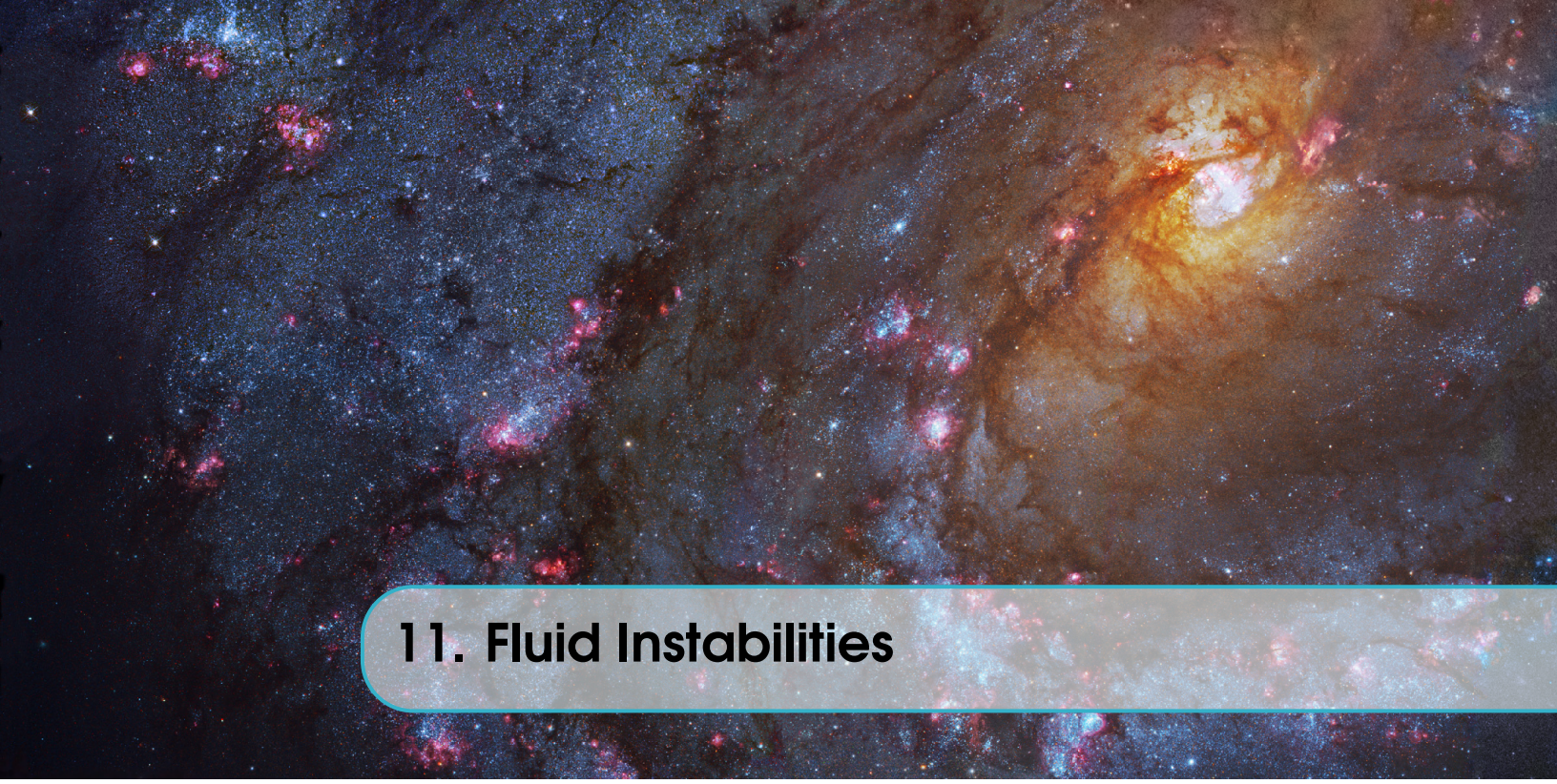
which leads (after some algebra) to

$$E \sim \frac{3\gamma-1}{2(\gamma+1)} \frac{4\pi}{3} \rho_0 R_{\max}^3 \frac{c_0^2}{\gamma(\gamma-1)}.$$

This generally works out to ~ 100pc for most supernovae.

10.5 Astrophysical Context

- The phases of shock breakout, the snowplow phase, sedov phase, and the dissipation phase.



11. Fluid Instabilities

We again use linear perturbation theory. We look for stability and instability in the perturbative solution. In the wave solutions, we studied stability. Now we look at instability. We want to know when they happen and what causes them.

11.1 Gravitational Instability

A self-gravitating fluid may become unstable and collapse under its own gravity. This phenomenon, known in the spherically symmetric case as the **Jeans instability**, determines the critical conditions under which small perturbations grow instead of oscillating. We analyze this behavior using **linear perturbation theory** applied to the fluid and Poisson equations. In this section, we'll look at the **Jeans Instability** as well as the scenario for **disks**, where we will be introduced to the **Toorme Q parameter**.

11.1.1 The Jeans Instability

Consider an infinite, homogeneous, and static medium characterized by a density ρ_0, pressure P_0, and gravitational potential Φ_0. In the unperturbed state,

$$\mathbf{u}_0 = 0, \qquad \rho = \rho_0 = \text{const}, \qquad P = P_0 = \text{const}. \tag{11.1}$$

The background potential satisfies Poissons equation,

$$\nabla^2 \Phi_0 = 4\pi G \rho_0. \tag{11.2}$$

R A strictly uniform, infinite medium cannot simultaneously satisfy Poissons equation and hydrostatic equilibrium; the potential Φ_0 would diverge. Following the **Jeans swindle**, we therefore neglect the self-gravity of the uniform background and consider only the gravitational response due to perturbations. Although formally inconsistent, this assumption correctly captures the physics of small-scale gravitational instability.

We introduce small perturbations about the background state:

$$\rho = \rho_0 + \delta\rho, \qquad P = P_0 + \delta P, \qquad \Phi = \Phi_0 + \delta\Phi, \qquad \mathbf{u} = \delta\mathbf{u}, \tag{11.3}$$

with $|\delta\rho|, |\delta P|, |\delta\Phi| \ll$ background quantities. Linearizing the governing equations yields:

Continuity:
$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{u} = 0, \tag{11.4}$$

Momentum:
$$\rho_0 \frac{\partial \delta\mathbf{u}}{\partial t} = -\nabla \delta P - \rho_0 \nabla \delta\Phi, \tag{11.5}$$

Poisson:
$$\nabla^2 \delta\Phi = 4\pi G\, \delta\rho. \tag{11.6}$$

For adiabatic perturbations, the pressure and density are related by

$$\delta P = c_s^2 \delta\rho, \tag{11.7}$$

where c_s is the adiabatic sound speed.

Derivation of the Wave Equation

Taking the time derivative of the continuity equation (11.4),

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \delta\mathbf{u}}{\partial t} = 0. \tag{11.8}$$

From the momentum equation (11.5) and the equation of state,

$$\frac{\partial \delta\mathbf{u}}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla \delta\rho - \nabla \delta\Phi. \tag{11.9}$$

Taking the divergence of (11.9) and using Poissons equation (11.6),

$$\nabla \cdot \frac{\partial \delta\mathbf{u}}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla^2 \delta\rho - 4\pi G\, \delta\rho. \tag{11.10}$$

Substituting (11.10) into (11.8) gives the closed-form **wave equation** for the density perturbation:

$$\boxed{\frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \nabla^2 \delta\rho + 4\pi G \rho_0\, \delta\rho.} \tag{11.11}$$

Dispersion Relation

We seek plane-wave solutions of the form

$$\delta\rho,\ \delta P,\ \delta\Phi,\ \delta\mathbf{u} \propto e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{11.12}$$

Substituting this ansatz into equation (11.11) yields

$$-\omega^2 \delta\rho = -c_s^2 k^2 \delta\rho + 4\pi G \rho_0\, \delta\rho, \tag{11.13}$$

or equivalently the **dispersion relation**

$$\boxed{\omega^2 = c_s^2 k^2 - 4\pi G \rho_0.} \tag{11.14}$$

The Jeans Criterion

Instability requires $\omega^2 < 0$, i.e.

$$c_s^2 k^2 < 4\pi G\rho_0. \tag{11.15}$$

Hence:

- For $c_s^2 k^2 > 4\pi G\rho_0$, we have $\omega^2 > 0$: perturbations oscillate as stable **sound waves**.
- For $c_s^2 k^2 < 4\pi G\rho_0$, we have $\omega^2 < 0$: perturbations grow exponentially, $\delta\rho \propto e^{|\omega|t}$ this is the **Jeans instability**.

The critical wavenumber, at which $\omega = 0$, defines the **Jeans wavenumber**:

$$k_J = \sqrt{\frac{4\pi G\rho_0}{c_s^2}}. \tag{11.16}$$

The corresponding **Jeans length** is

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}}. \tag{11.17}$$

Perturbations with wavelengths $\lambda > \lambda_J$ are gravitationally unstable and will collapse, while smaller perturbations are stabilized by pressure support.

Physical Interpretation and Timescales

The Jeans criterion can be understood in terms of competing timescales:

$$t_{\rm ff} \sim \frac{1}{\sqrt{G\rho_0}} \qquad \text{(free-fall timescale)}, \tag{11.18}$$

$$t_{\rm sound} \sim \frac{1}{kc_s} \qquad \text{(sound-crossing timescale)}. \tag{11.19}$$

Gravitational collapse occurs when the free-fall time is shorter than the sound-crossing time,

$$t_{\rm ff} < t_{\rm sound} \quad \Longleftrightarrow \quad k < k_J. \tag{11.20}$$

This effectively means that the system cannot respond to infall compression fast enough to reverse the collapse.

11.1.2 Disk Instability and Toomre's Q Parameter

So far, we have examined the stability of a **homogeneous, isotropic medium** and found that a self-gravitating fluid becomes unstable above the Jeans length. However, many astrophysical systems of interestsuch as galactic or accretion disksare **rotationally supported, flattened structures**. In these systems, rotation and shear play crucial roles in stabilizing against self-gravity. To analyze such systems, we again turn to **linear perturbation theory**, but this time we work in a *rotating frame* appropriate to the disk geometry. Consider

a thin, axisymmetric disk with angular velocity profile $\Omega(R)$ and unperturbed velocity field

$$\mathbf{u}_0 = R\,\Omega(R)\,\hat{\boldsymbol{\phi}}.$$

The disk has an unperturbed surface density Σ_0 and corresponding vertically integrated pressure P_0. We are interested in small, axisymmetric perturbations to this background configuration.

Definition 11.1.1 — The Shearing Sheet Approximation. We analyze the dynamics in a local Cartesian patch centered at radius R_0 that co-rotates with the disk. In this local frame, x points radially outward and y lies in the azimuthal direction. The differential rotation of the disk appears as a linear shear:

$$\mathbf{u}_0 \simeq (0,\, -q\,\Omega_0\,x), \qquad q \;\equiv\; -\frac{d\ln\Omega}{d\ln R}\Big|_{R_0}.$$

This is called the **shearing sheet** approximation. It captures the local effects of rotation and shear while neglecting global curvature terms, allowing the use of plane-wave perturbations

$$\propto e^{i(kx-\omega t)}.$$

The approximation is valid when the perturbation wavelength is much smaller than the global scale of the disk: $\lambda \ll R_0$.

We can now introduce the notion of the **epicycle frequency**, which is going to be of chief importance to use as we continue this derivation. When a fluid element in a rotating disk is displaced slightly in radius, it no longer moves at the circular velocity appropriate to its new position. As a result, the imbalance between centrifugal and gravitational forces causes it to execute small oscillations about its equilibrium orbit. The frequency of these oscillations is called the ***epicyclic frequency***.

Consider an element initially on a circular orbit at radius R_0 with angular velocity Ω_0. Let it be displaced radially by a small amount $x \ll R_0$ while keeping its specific angular momentum $L = R^2\dot{\phi}$ approximately conserved. The effective potential for radial motion in the plane of the disk is

$$\Phi_{\rm eff}(R) = \Phi(R) + \frac{L^2}{2R^2},$$

where $\Phi(R)$ is the gravitational potential. Expanding $\Phi_{\rm eff}$ to first order about R_0 gives the equilibrium condition for circular motion,

$$\frac{d\Phi_{\rm eff}}{dR}\bigg|_{R_0} = 0 \quad \Rightarrow \quad \frac{v_0^2}{R_0} = \frac{d\Phi}{dR}\bigg|_{R_0},$$

and to second order we find the restoring force per unit mass:

$$\frac{d^2\Phi_{\rm eff}}{dR^2}\bigg|_{R_0} \;=\; \frac{d^2\Phi}{dR^2}\Big|_{R_0} + 3\Omega_0^2.$$

The radial equation of motion for small displacements $x(t)$ about R_0 is therefore

$$\ddot{x} = -\frac{d^2\Phi_{\rm eff}}{dR^2}\bigg|_{R_0} x \;=\; -\kappa^2\,x,$$

which is a simple harmonic oscillator with frequency

$$\boxed{\kappa^2 \;\equiv\; R\frac{d\Omega^2}{dR} + 4\Omega^2 \;=\; 4\Omega^2 + 2R\,\Omega\,\frac{d\Omega}{dR} \;=\; 2(2-q)\Omega^2, \qquad q \equiv -\frac{d\ln\Omega}{d\ln R}.}$$

Thus κ quantifies the restoring strength of differential rotation: it measures how rapidly a displaced parcel oscillates around its circular orbit.

- For a **Keplerian disk**, $\Omega \propto R^{-3/2}$, giving $\kappa = \Omega$.
- For a **flat rotation curve**, $\Omega \propto R^{-1}$, giving $\kappa = \sqrt{2}\,\Omega$.

In both cases, the nonzero value of κ reflects the stabilizing role of rotation and shear in opposing local gravitational collapse. We now proceed to derive the dispersion relation for small perturbations in a self-gravitating, differentially rotating, thin gaseous disk.

Linearized Equations

Let $\Sigma = \Sigma_0 + \delta\Sigma$ be the perturbed surface density, $P = P_0 + \delta P$ the perturbed pressure, and $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$ the velocity field. For an isothermal or adiabatic disk, we use the effective 2D equation of state

$$\delta P = c_s^2\,\delta\Sigma,$$

where c_s is the sound speed.

In the local co-rotating frame, the linearized hydrodynamic equations are:

$$\begin{aligned}
&\text{Continuity:} && \frac{\partial\delta\Sigma}{\partial t} + \Sigma_0\nabla\cdot\delta\mathbf{u} = 0,\\
&\text{Radial momentum:} && \frac{\partial\delta u_x}{\partial t} - 2\Omega_0\,\delta u_y = -\frac{\partial}{\partial x}\,(\delta h + \delta\Phi),\\
&\text{Azimuthal momentum:} && \frac{\partial\delta u_y}{\partial t} + \frac{\kappa^2}{2\Omega_0}\,\delta u_x = 0,
\end{aligned}$$

where $\delta h \equiv \delta P/\Sigma_0 = c_s^2\,\delta\Sigma/\Sigma_0$, and $\delta\Phi$ is the perturbation to the gravitational potential. The statement for the momentum equation comes directly from the fully general **Euler Equation** in the form

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\Sigma} - \nabla\Phi - 2\boldsymbol{\Omega}\times\mathbf{u} + \Omega^2\mathbf{r}.$$

At first order, the gravitational force directly balances the centrifugal force and therefore eliminates centrifugal force from the equation. We get the κ from the gradients of the potential.

For a razor-thin disk, all the mass is confined to the midplane ($z = 0$), so perturbations in the surface density $\delta\Sigma(\mathbf{r})$ act as a 2D source for the gravitational potential. Starting from Poissons equation,

$$\nabla^2\,\delta\Phi(\mathbf{r}, z) = 4\pi G\,\delta\rho(\mathbf{r}, z),$$

the perturbed density may be written as

$$\delta\rho(\mathbf{r}, z) = \delta\Sigma(\mathbf{r})\,\delta(z),$$

where $\delta(z)$ enforces confinement to the midplane.

Taking a 2D Fourier transform in the disk plane and solving for the potential at $z = 0$, one finds

$$\boxed{\delta\Phi(\mathbf{k}) \;=\; -\frac{2\pi G}{|\mathbf{k}|}\,\delta\Sigma(\mathbf{k})} \tag{11.21}$$

Really we're integrating over a razor thin region of the disk

This relation is the Fourier-space Greens function for a razor-thin sheet. The factor of $1/|\mathbf{k}|$ reflects the long-range nature of gravity: large-scale modes ($|\mathbf{k}| \to 0$) produce stronger potentials, while the minus sign ensures the potential is attractive. The exponential $e^{-|k||z|}$ vertical dependence has already been evaluated at the midplane ($z = 0$).

Equation (11.21) is essential for disk stability analysis: it provides the link between density perturbations and their self-gravitational potential, which competes with pressure and rotation in the Toomre stability criterion. We take all perturbations to have the WKB form

$$\delta\Sigma,\ \delta\Phi,\ \delta u_x,\ \delta u_y \;\propto\; e^{i(kx-\omega t)}.$$

The continuity equation then gives

$$-i\omega\,\delta\Sigma + ik\Sigma_0\,\delta u_x = 0 \quad\Longrightarrow\quad \delta\Sigma = \frac{\Sigma_0 k}{\omega}\,\delta u_x.$$

The azimuthal momentum equation provides

$$-i\omega\,\delta u_y + \frac{\kappa^2}{2\Omega_0}\,\delta u_x = 0 \quad\Longrightarrow\quad \delta u_y = i\,\frac{\kappa^2}{2\Omega_0\omega}\,\delta u_x.$$

Substituting these results and (11.21) into the radial momentum equation gives

$$-i\omega\,\delta u_x - i\frac{\kappa^2}{\omega}\,\delta u_x = -i\frac{k^2}{\omega}\,\delta u_x\left(c_s^2 - \frac{2\pi G\Sigma_0}{|k|}\right).$$

Dividing through by common factors yields the **dispersion relation**:

$$\boxed{\omega^2 = \kappa^2 - 2\pi G\Sigma_0|k| + c_s^2k^2.} \tag{11.22}$$

Stability Criterion and Toomre's Q Parameter

Equation (11.22) shows that rotation (κ^2) and pressure ($c_s^2k^2$) act as stabilizing influences, while self-gravity ($-2\pi G\Sigma_0|k|$) drives collapse. To find the condition for stability, we minimize $\omega^2(k)$ with respect to k. The most unstable wavenumber satisfies

$$\frac{d\omega^2}{dk} = 0 \quad\Rightarrow\quad k_{\rm crit} = \frac{\pi G\Sigma_0}{c_s^2}.$$

At this k, the minimum value of ω^2 is

$$\omega^2_{\rm min} = \kappa^2 - \frac{(\pi G\Sigma_0)^2}{c_s^2}.$$

For the disk to be stable for all wavelengths, we require $\omega_{\min}^2 > 0$, giving the **Toomre stability criterion**:

$$\boxed{Q \equiv \frac{c_s \kappa}{\pi G \Sigma_0} > 1.} \tag{11.23}$$

If $Q < 1$, some wavenumbers yield $\omega^2 < 0$, and the corresponding perturbations grow exponentiallyindicating **gravitational instability**.

Physical Interpretation

The Toomre parameter quantifies the competition between three effects:

$$Q = \frac{\text{pressure support } (c_s) \times \text{rotational support } (\kappa)}{\text{self-gravity } (\pi G \Sigma_0)}.$$

- For $Q > 1$, the combination of pressure and rotational shear stabilizes the disk.
- For $Q < 1$, gravity overwhelms these effects, leading to fragmentation or ring-like collapse.

The most unstable (fastest-growing) wavelength is obtained by substituting k_{crit}:

$$\lambda_{\text{crit}} = \frac{2c_s^2}{G\Sigma_0}. \tag{11.24}$$

Perturbations with $\lambda < \lambda_{\text{crit}}$ are stabilized by pressure, while those with $\lambda > \lambda_{\text{crit}}$ are stabilized by shear. Instability is confined to scales near λ_{crit}, where self-gravity dominates both effects.

11.2 Interface Instabilities

We consider two inviscid, incompressible, irrotational fluids separated by a planar interface at $z = 0$. The lower fluid (region $z < 0$) has density ρ_0 and uniform horizontal velocity $U_0\,\hat{\mathbf{x}}$; the upper fluid (region $z > 0$) has density ρ_1 and uniform horizontal velocity $U_1\,\hat{\mathbf{x}}$. A constant gravitational acceleration $\mathbf{g} = -g\,\hat{\mathbf{z}}$ acts downward. We allow for surface tension σ (set $\sigma = 0$ to recover the purely gravitational case).

Because the base state is incompressible and irrotational, we introduce velocity potentials Ψ_i such that

$$\mathbf{u}_i = \nabla \Psi_i$$

and

$$\nabla^2 \Psi_i = 0$$

in each layer $i \in \{0, 1\}$. In the unperturbed state, in the uperturbed state, we have a simple integration to find that

$$\Psi_i = u_i x.$$

Applying the **Bernoulli Theorem** in the case of an **unsteady flow**, yields

$$\frac{\partial \Psi_i}{\partial t} + \frac{1}{2}\left|\nabla \Psi_i\right|^2 + \frac{P_i}{\rho_i} + \Phi = F(t), \tag{11.25}$$

where $\Phi = gz$ is the gravitational potential and $F(t)$ is a spatially uniform Bernoulli function.

Now that we have set up the equilibrated scenario, let's start trying to derive the behavior of a perturbation to the surface. We perturb the interface to $z = \xi(x,t)$ with $|\xi| \ll 1$, and write

$$\Psi_i^{(1)} = \Psi_i^{(0)} + \delta\psi_i = U_i x + \delta\psi_i, \qquad P_i^{(1)} = P_i^{(0)} + \delta P_i,$$

where $\delta\Psi_i, \delta P_i$ are first order linear perturbations about the mean. Since we still maintain **incompressibility and irrotational flow**, we have

$$\nabla^2 \delta\psi_i = 0.$$

This can be made formal by invoking the Kelvin Circulation Theorem. We adopt a normal-mode (plane-wave) ansatz

$$\{\phi_i,\, p_i,\, \xi\} \propto e^{\,i(kx-\omega t)}, \tag{11.26}$$

and require decay away from the interface:

$$\phi_0(x,z,t) = A_0\, e^{\,kz}\, e^{\,i(kx-\omega t)}, \qquad z < 0, \tag{11.27}$$

$$\phi_1(x,z,t) = A_1\, e^{-kz}\, e^{\,i(kx-\omega t)}, \qquad z > 0, \tag{11.28}$$

so that $\partial_z\phi_0 \to 0$ as $z \to -\infty$ and $\partial_z\phi_1 \to 0$ as $z \to +\infty$. This allows us to provide a self-consistent solution to the vertical Laplace equation without allowing divergence of the flow field. Our task now is to understand the structure of this solution and its implications for phenomenology.

Boundary Conditions

As is typical in perturbation problems, we can explore relevant boundary conditions to obtain consistency conditions for our ansatz. Let's begin by looking at the interface, at which there will be both a **kinematic condition** describing the velocities and also a **traction condition** determining the pressures.

The Kinematic Condition

Since the interface is a so-called **free-interface**, we cannot allow a particle on the surface to leave the surface. Thus, the vertical velocity of a particle u_z must be related to the perturbed surface such that (to linear order)

$$\frac{D\xi}{Dt} = \partial_t\xi + u_{ix}\partial_x\xi = \partial_z\delta\psi_i.$$

The idea here is that we have a vertical displacement which must match the change in the surface for the particle. We get the fluid velocity from the z derivative of the perturbed potential and from the surface. With the ansatz (11.26) this relationship yields for each surface that

$$-i(\omega - kU_0)\,\xi = kA_0, \qquad -i(\omega - kU_1)\,\xi = -kA_1, \tag{11.29}$$

so that

$$A_0 = -\frac{i}{k}(\omega - kU_0)\,\xi, \qquad A_1 = \frac{i}{k}(\omega - kU_1)\,\xi. \tag{11.30}$$

The Traction (Dynamic) Condition

At the interface, the **Cauchy traction balance law** requires that the jump in normal stress equals the restoring force due to surface tension. For an inviscid, isotropic fluid, the only stress contribution is the scalar pressure, so we may write

$$[\,p\,]_1^0 \;\equiv\; p_1 - p_0 \;=\; -\,\sigma\,\nabla_s\!\cdot\!\hat{\mathbf{n}}, \tag{11.31}$$

where $\hat{\mathbf{n}}$ is the local unit normal to the interface and $\nabla_s\!\cdot\!\hat{\mathbf{n}}$ is the mean curvature. For a weakly perturbed interface of the form $z = \xi(x,t)$, where $|\xi| \ll 1$, we may linearize the curvature as

$$\nabla_s\!\cdot\!\hat{\mathbf{n}} \;\approx\; -\,\partial_{xx}\xi.$$

Substituting this into (11.31) gives

$$p_1 - p_0 \;=\; \sigma\,\partial_{xx}\xi \;=\; -\sigma k^2\,\xi, \tag{11.32}$$

where we have used $\partial_{xx}\xi = -k^2\xi$ under the plane-wave ansatz. **This expresses the tendency of surface tension to resist curvature (short-wavelength) perturbations.**

To relate the pressure perturbations to the perturbed potentials, we now linearize the **unsteady Bernoulli equation** (11.25) about the background flow. Starting from

$$\frac{\partial \Psi_i}{\partial t} + \frac{1}{2}\,|\nabla\Psi_i|^2 + \frac{P_i}{\rho_i} + \Phi = F(t), \tag{11.33}$$

we substitute in the perturbed quantities,

$$\Psi_i = \Psi_i^{(0)} + \delta\psi_i, \qquad P_i = P_i^{(0)} + \delta P_i.$$

Expanding to first order, we find

$$\frac{\partial}{\partial t}\left(\Psi_i^{(0)} + \delta\psi_i\right) + \frac{1}{2}\left|\nabla(\Psi_i^{(0)} + \delta\psi_i)\right|^2 + \frac{P_i^{(0)} + \delta P_i}{\rho_i} + \Phi = F(t).$$

Separating equilibrium and perturbation terms, and noting that the background flow $(\Psi_i^{(0)}, P_i^{(0)})$ already satisfies Bernoullis equation, we retain only the linear perturbations:

$$\frac{\partial \delta\psi_i}{\partial t} + \nabla\Psi_i^{(0)}\!\cdot\!\nabla\delta\psi_i + \frac{\delta P_i}{\rho_i} = 0.$$

Evaluating at the Displaced Interface

The difficulty with the above expression is that we really know $P_i^{(0)}$ at the $z = 0$ interface, so it is not straightforward to apply the traction balance condition directly at the displaced surface $z = \xi(x,t)$. To address this, we use a Taylor expansion to relate the pressure field at the moving interface to that at the reference plane $z = 0$. Expanding the total pressure to first order gives

$$P_i(z=\xi) \;\approx\; P_i(0) + \xi\,\left.\frac{\partial P_i}{\partial z}\right|_0.$$

We now separate this into equilibrium and perturbation components:

$$P_i^{(0)}(z=\xi) + \delta P_i(z=\xi) \;\approx\; \left[P_i^{(0)}(0) + \delta P_i(0)\right] + \xi \left.\frac{\partial P_i^{(0)}}{\partial z}\right|_0 .$$

Subtracting the equilibrium balance at $z = 0$, the pressure perturbation at the displaced interface becomes

$$\delta P_i(z=\xi) = \delta P_i(z=0) - \xi \left.\frac{\partial P_i^{(0)}}{\partial z}\right|_0 . \tag{11.34}$$

From hydrostatic equilibrium,

$$\frac{\partial P_i^{(0)}}{\partial z} = -\,\rho_i g,$$

so that

$$\delta P_i(z=\xi) = \delta P_i(z=0) + \rho_i g\,\xi. \tag{11.35}$$

In other words, a fluid parcel displaced upward by ξ experiences a lower local pressure at the new interface, as expected from hydrostatic balance. We now use this to evaluate the Bernoulli relation at the displaced boundary. At $z = 0$, the linearized Bernoulli equation is

$$\frac{\delta P_i(z=0)}{\rho_i} = -\left(\frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x}\right)\delta\psi_i.$$

Using the propagation relation (11.35) to move this result up to $z = \xi$, we obtain

$$\frac{\delta P_i(z=\xi)}{\rho_i} = -\left(\frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x}\right)\delta\psi_i - g\,\xi.$$

This is the form we will use at the boundary for the traction balance.

Using the plane-wave ansatz $\delta\psi_i = A_i e^{\pm kz} e^{i(kx-\omega t)}$, evaluated at $z = 0$, gives

$$\left(\frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x}\right)\delta\psi_i = -i(\omega - kU_i)\,A_i\,e^{\,i(kx-\omega t)}.$$

Substituting this into the previous expression yields

$$\delta P_i(z=\xi) = \rho_i\left[i(\omega - kU_i)A_i - g\xi\right]e^{\,i(kx-\omega t)}. \tag{11.36}$$

We now substitute the amplitudes A_i from the kinematic condition

$$A_0 = -\frac{i}{k}(\omega - kU_0)\xi, \qquad A_1 = \frac{i}{k}(\omega - kU_1)\xi,$$

to obtain

$$\delta P_0 = \rho_0\left[\frac{(\omega - kU_0)^2}{k} - g\right]\xi\,e^{i(kx-\omega t)}, \tag{11.37}$$

$$\delta P_1 = -\,\rho_1\left[\frac{(\omega - kU_1)^2}{k} + g\right]\xi\,e^{i(kx-\omega t)}. \tag{11.38}$$

The negative sign for δP_1 arises because $\partial_z \phi_1 = -kA_1$ at $z = 0$, reversing the vertical velocity direction above the interface.

Finally, we apply the traction balance condition (11.32) at $z = 0$,

$$\delta P_1 - \delta P_0 = -\sigma k^2 \xi.$$

Substituting the above expressions for δP_i and canceling the common factor $\xi e^{i(kx-\omega t)}$, we obtain after rearranging:

$$\rho_0 (\omega - kU_0)^2 + \rho_1 (\omega - kU_1)^2 \;=\; (\rho_0 - \rho_1)\, g\, k + \sigma k^3. \tag{11.39}$$

This is the canonical dispersion relation for a two-layer, inviscid, incompressible interface under gravity and surface tension, derived by relating the traction balance at the displaced interface to pressures referenced at $z = 0$.

Solving for ω

Equation (11.39) is quadratic in ω. Solving explicitly yields

$$\boxed{\omega \;=\; k\,\bar{U} \;\pm\; \sqrt{-\frac{\rho_0 \rho_1}{(\rho_0 + \rho_1)^2} (\Delta U)^2 k^2 \;+\; \frac{\rho_0 - \rho_1}{\rho_0 + \rho_1} g\, k \;+\; \frac{\sigma}{\rho_0 + \rho_1} k^3},} \tag{11.40}$$

where

$$\bar{U} \equiv \frac{\rho_0 U_0 + \rho_1 U_1}{\rho_0 + \rho_1}, \qquad \Delta U \equiv U_1 - U_0.$$

The term under the square root determines the stability of the interface: it is positive for oscillatory (stable) motion and negative for exponential (unstable) growth. The relative magnitudes of the shear term, gravity term, and surface tension term determine whether the instability is of the Kelvin–Helmholtz or Rayleigh–Taylor type.

11.2.1 Rayleigh–Taylor Instability

The **Rayleigh–Taylor instability (RTI)** describes the situation in which a heavy fluid lies above a lighter one in a gravitational field. To isolate this effect, **we neglect both surface tension and shear** by setting $\sigma = 0$ and $U_0 = U_1 = 0$. The dispersion relation (11.39) then simplifies to

$$\omega^2 = -\frac{\rho_1 - \rho_0}{\rho_0 + \rho_1} g\, k. \tag{11.41}$$

If the **heavier fluid is below the lighter fluid** ($\rho_0 > \rho_1$), then $\rho_1 - \rho_0 < 0$ and $\omega^2 > 0$, giving

$$\omega = \pm \sqrt{\frac{\rho_0 - \rho_1}{\rho_0 + \rho_1} g\, k}.$$

The system oscillates with a real frequency ω, corresponding to small-amplitude gravity waves at the interface. **If the heavier fluid is *above* the lighter one** ($\rho_1 > \rho_0$), then $\omega^2 < 0$ and the growth rate becomes imaginary:

$$\gamma = \sqrt{\frac{\rho_1 - \rho_0}{\rho_0 + \rho_1} g\, k}. \tag{11.42}$$

In this case, perturbations grow exponentially as $e^{\gamma t}$, forming the characteristic "mushroom" plumes of the Rayleigh–Taylor instability.

Surface Tension

If surface tension is present, small wavelengths are stabilized. Including σ in the dispersion relation gives

$$\omega^2 = -\frac{\rho_1 - \rho_0}{\rho_0 + \rho_1} g\, k + \frac{\sigma}{\rho_0 + \rho_1} k^3.$$

Setting $\omega^2 = 0$ **defines the *critical wavenumber* separating stable and unstable modes:**

$$\boxed{k_c = \frac{(\rho_1 - \rho_0)\, g}{\sigma}.} \tag{11.43}$$

Modes with $k > k_c$ (short wavelengths) are stabilized by surface tension, while long waves ($k < k_c$) remain unstable.

Big Idea

Rayleigh–Taylor Instability When a **heavier fluid lies above a lighter fluid** in a gravitational field, any small displacement of the interface causes the heavy fluid to sink and the light fluid to rise, further amplifying the initial disturbance. This leads to **exponential growth** of perturbations with growth rate

$$\gamma_{\mathrm{RT}} = \sqrt{\frac{\rho_1 - \rho_0}{\rho_0 + \rho_1} g\, k},$$

and produces characteristic "spike-and-bubble" structures as the two fluids mix. Gravity provides the energy source, while surface tension and viscosity act to stabilize short-wavelength modes. The RT instability is fundamentally driven by the conversion of gravitational potential energy into kinetic motion.

11.2.2 Kelvin-Helmholtz Instability

The **Kelvin–Helmholtz instability (KHI)** arises from shear between two fluid layers with different velocities. It is a direct consequence of the results derived above. If we examine the general dispersion relation, we have

$$\omega = k\bar{U} \pm \sqrt{-\frac{\rho_0\rho_1}{(\rho_0 + \rho_1)^2} (\Delta U)^2 k^2 + \frac{\rho_0 - \rho_1}{\rho_0 + \rho_1} g\, k + \frac{\sigma}{\rho_0 + \rho_1} k^3}. \tag{11.44}$$

For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the quantity inside the square root must be negative. This gives the general **instability condition**

$$\frac{\rho_0 - \rho_1}{\rho_0 + \rho_1} g\, k + \frac{\sigma}{\rho_0 + \rho_1} k^3 < \frac{\rho_0\rho_1}{(\rho_0 + \rho_1)^2} (\Delta U)^2\, k^2. \tag{11.45}$$

From this, we can immediately see that the gravity term (proportional to gk) tends to stabilize long-wavelength modes, while the surface tension term (proportional to k^3) stabilizes short-wavelength modes. The velocity shear term, scaling as $(\Delta U)^2 k^2$, acts as the destabilizing mechanism. We now examine the three most instructive limiting cases:

Case 1: No Gravity, No Surface Tension ($g = 0$, $\sigma = 0$)
In this case, the dispersion relation simplifies to

$$\rho_0(\omega - kU_0)^2 + \rho_1(\omega - kU_1)^2 = 0, \tag{11.46}$$

which yields the solution

$$\omega = k\bar{U} \pm i\frac{\sqrt{\rho_0\rho_1}}{\rho_0 + \rho_1}\,|k\,\Delta U|. \tag{11.47}$$

The imaginary part of ω implies an exponential growth rate

$$\gamma = \frac{\sqrt{\rho_0\rho_1}}{\rho_0 + \rho_1}\,|k\,\Delta U|. \tag{11.48}$$

Since $\gamma > 0$ for any nonzero shear ΔU, the interface is **unstable for all wavelengths**. Furthermore, $\gamma \propto |k|$, so shorter wavelengths grow faster. In realistic fluids, viscosity or finite interface thickness imposes a small-scale cutoff, preventing unbounded growth. This limit represents the classical, idealized Kelvin–Helmholtz instability: *any finite velocity shear leads to instability.*

Case 2: No Gravity, Finite Surface Tension ($g = 0$, $\sigma \neq 0$)
When surface tension is present, the dispersion relation becomes

$$\omega = k\bar{U} \pm \sqrt{-\frac{\rho_0\rho_1}{(\rho_0 + \rho_1)^2}\,(\Delta U)^2k^2 + \frac{\sigma}{\rho_0 + \rho_1}\,k^3}. \tag{11.49}$$

The instability condition (11.45) reduces to

$$\frac{\sigma}{\rho_0 + \rho_1}\,k^3 < \frac{\rho_0\rho_1}{(\rho_0 + \rho_1)^2}\,(\Delta U)^2\,k^2, \tag{11.50}$$

or equivalently,

$$k < k_{\text{crit}} = \frac{\rho_0\rho_1(\Delta U)^2}{\sigma(\rho_0 + \rho_1)}. \tag{11.51}$$

Thus, only perturbations with wavenumber $k < k_{\text{crit}}$ (i.e., long wavelengths) are unstable, while short-wavelength modes ($k > k_{\text{crit}}$) are stabilized by surface tension. Physically, surface tension acts to smooth out small undulations at the interface by providing a restoring force that increases with curvature. Therefore, surface tension defines a **maximum unstable wavenumber** beyond which the instability is suppressed.

Case 3: Finite Gravity, No Surface Tension ($g \neq 0$, $\sigma = 0$)
In the absence of surface tension, gravity introduces a stabilizing (or destabilizing) term depending on the orientation of the density gradient. The dispersion relation becomes

$$\omega = k\bar{U} \pm \sqrt{-\frac{\rho_0\rho_1}{(\rho_0 + \rho_1)^2}\,(\Delta U)^2k^2 + \frac{\rho_0 - \rho_1}{\rho_0 + \rho_1}\,g\,k}. \tag{11.52}$$

The instability criterion (11.45) gives

$$\frac{\rho_0 - \rho_1}{\rho_0 + \rho_1}\,g\,k < \frac{\rho_0\rho_1}{(\rho_0 + \rho_1)^2}\,(\Delta U)^2\,k^2. \tag{11.53}$$

If the heavy fluid lies **below** the lighter one ($\rho_0 > \rho_1$, $g > 0$), the first term is positive and acts to stabilize long-wavelength perturbations. Setting the two terms equal gives the critical wavenumber

$$k_{\text{crit}} = \frac{(\rho_0 - \rho_1)(\rho_0 + \rho_1)g}{\rho_0 \rho_1 (\Delta U)^2}. \tag{11.54}$$

For $k < k_{\text{crit}}$, gravity dominates and the system is stable, while for $k > k_{\text{crit}}$, velocity shear dominates and instability sets in. If, however, the heavy fluid lies **above** the light one ($\rho_0 < \rho_1$), the gravitational term becomes destabilizing, and the system is **always unstable**, even without shearthis is the classic **Rayleigh–Taylor instability** limit.

Big Idea

Kelvin–Helmholtz Instability The Kelvin–Helmholtz instability occurs whenever two fluid layers move past each other with a velocity shear across their interface. In the inviscid, tensionless, and gravity-free limit, the growth rate is

$$\gamma_{\text{KH}} = \frac{\sqrt{\rho_0 \rho_1}}{\rho_0 + \rho_1} |k\, \Delta U|,$$

so any finite shear is unstable at all wavelengths. Gravity stabilizes long wavelengths when the heavy fluid lies below, and surface tension stabilizes short wavelengths by damping curvature. In astrophysical and geophysical contexts, KHI governs the onset of shear-driven mixing, vorticity generation, and turbulent entrainment between stratified layers, jets, and boundary shears.

11.3 Thermal Instability

In addition to dynamical instabilities driven by gravity or shear, fluids may also become unstable due to **thermodynamic imbalances**. These occur when the local heating and cooling processes of a gas act to amplify small temperature or density perturbations rather than damp them. The resulting runaway condensation or rarefaction leads to the formation of a **multi-phase medium**. This class of phenomena is known collectively as **thermal instability** Field [1965], and is particularly important in astrophysical contexts such as the *interstellar medium*, *accretion flows*, and *intra-cluster gas*.

In a system where radiative heating and cooling are both present, thermal equilibrium is maintained when the local heating rate per unit mass Γ balances the cooling rate Λ:

$$\dot{Q} \equiv \Gamma - \Lambda = 0. \tag{11.55}$$

If a small perturbation increases the temperature or density, the corresponding change in $\dot{Q}$ determines whether the disturbance grows or decays. If heating dominates in response to a cooling perturbation, the system is stable; but if cooling increases faster than pressure can respond, the perturbation grows exponentially. In some sense, we heuristically think that if

$$\frac{d\Lambda}{dT} < 0,$$

then as we increase temperature, we cool less efficiently and we can lead to runaway condensation. In the next section, we'll be more formal about this assertion.

Governing Equations

We begin from the standard equations of hydrodynamics for a compressible, inviscid fluid with a general energy source term $\dot{Q}$ (energy gained per unit mass per unit time):

$$\text{Continuity:} \qquad \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0, \tag{11.56}$$

$$\text{Momentum:} \qquad \rho \frac{d\mathbf{u}}{dt} = -\nabla P, \tag{11.57}$$

$$\text{Energy:} \qquad T\frac{ds}{dt} = -\dot{Q}, \tag{11.58}$$

where s is the specific entropy and $\dot{Q} > 0$ corresponds to *net cooling*.

Equation (11.58) follows directly from the first law of thermodynamics,

$$du = T\,ds - P\,d\left(\frac{1}{\rho}\right),$$

so that

$$\frac{du}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right) - \dot{Q}.$$

We assume an ideal gas equation of state,

$$P = (\gamma - 1)\rho u = K\rho^{\gamma}, \tag{11.59}$$

where **the polytropic constant K encodes the specific entropy.** Differentiating this expression gives

$$\frac{dK}{dt} = \frac{1}{\rho^{\gamma}}\frac{dP}{dt} - \frac{\gamma P}{\rho^{\gamma+1}}\frac{d\rho}{dt}. \tag{11.60}$$

From the energy equation, we may equivalently write

$$\frac{dK}{dt} = -\frac{(\gamma - 1)}{\rho^{\gamma-1}}\,\dot{Q}. \tag{11.61}$$

Thus, entropy evolution and the associated thermodynamic stability are governed by how $\dot{Q}$ responds to perturbations in P and ρ.

Linear Perturbation Analysis

We consider a uniform, static equilibrium characterized by (ρ_0, P_0, T_0) satisfying

$$\dot{Q}_0(\rho_0, T_0) = 0.$$

We introduce small perturbations of the form

$$\rho = \rho_0 + \delta\rho, \quad P = P_0 + \delta P, \quad T = T_0 + \delta T, \quad \mathbf{u} = \delta\mathbf{u}, \tag{11.62}$$

and assume normal-mode solutions $\propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Neglecting self-gravity and viscosity, the linearized equations become

$$-i\omega\,\delta\rho + i\rho_0\mathbf{k}\cdot\delta\mathbf{u} = 0, \tag{11.63}$$

$$-i\omega\rho_0\,\delta\mathbf{u} = -i\mathbf{k}\,\delta P, \tag{11.64}$$

and from the energy equation,

$$-i\omega\,\frac{\delta P}{P_0} - i\gamma\omega\,\frac{\delta\rho}{\rho_0} = (\gamma-1)\frac{\delta\dot{Q}}{P_0}. \tag{11.65}$$

The perturbation to the cooling function is expanded to first order as

$$\delta\dot{Q} = \left(\frac{\partial\dot{Q}}{\partial\rho}\right)_T \delta\rho + \left(\frac{\partial\dot{Q}}{\partial T}\right)_\rho \delta T. \tag{11.66}$$

Using the ideal-gas relation $\delta P/P_0 = \delta\rho/\rho_0 + \delta T/T_0$, we can eliminate δT to express $\delta\dot{Q}$ in terms of δP and $\delta\rho$:

$$\delta\dot{Q} = A\,\delta\rho + B\,\delta P, \qquad \begin{cases} A = \left[\left(\frac{\partial\dot{Q}}{\partial\rho}\right)_T - \frac{\rho_0}{T_0}\left(\frac{\partial\dot{Q}}{\partial T}\right)_\rho\right], \\ B = \dfrac{\rho_0}{P_0}\left(\frac{\partial\dot{Q}}{\partial T}\right)_\rho. \end{cases} \tag{11.67}$$

Combining equations (11.63)(11.65), we eliminate $\delta\mathbf{u}$ and $\delta\dot{Q}$ to obtain a closed relation between δP and $\delta\rho$. From (11.64), $\delta\mathbf{u} = (\mathbf{k}/\omega\rho_0)\,\delta P$, and substitution into (11.63) gives

$$\omega^2\,\delta\rho = k^2\frac{\rho_0}{\gamma P_0}\,\delta P. \tag{11.68}$$

Substituting this and (11.66)(11.67) into (11.65) yields a cubic dispersion relation of the form

$$\boxed{\omega^3 + i(\gamma-1)\left[\frac{BP_0}{\rho_0} + \frac{A}{k^2c_s^2}\right]\omega^2 - c_s^2k^2\omega - i(\gamma-1)A = 0,} \tag{11.69}$$

where $c_s^2 = \gamma P_0/\rho_0$ is the adiabatic sound speed.

Limiting Cases and the Field Criterion

The cubic equation (11.69) admits three branches corresponding to different physical regimes:

- **Adiabatic acoustic modes:** When radiative effects are negligible ($A, B \to 0$), we recover the sound-wave relation $\omega^2 = c_s^2k^2$.
- **Isobaric mode:** For long wavelengths ($k \to 0$), pressure equilibrates rapidly, $\delta P \simeq 0$, and the instability evolves at nearly constant pressure. The growth rate simplifies to

$$\boxed{\omega_{\rm isoP} = -\,i\,\frac{(\gamma-1)}{P_0}\left(\frac{\partial\dot{Q}}{\partial T}\right)_P T_0.} \tag{11.70}$$

The system is unstable if the net cooling increases with temperature at constant pressure:

$$\left(\frac{\partial \dot{Q}}{\partial T}\right)_P > 0.$$

This is the classic **Field criterion for isobaric thermal instability**.

- **Isochoric mode:** If the density is effectively fixed (e.g. very slow pressure response), the growth rate becomes

$$\boxed{\omega_{\text{isoC}} = -\,i\,\frac{(\gamma-1)}{T_0}\left(\frac{\partial \dot{Q}}{\partial T}\right)_\rho}. \tag{11.71}$$

The gas is unstable if $\left(\frac{\partial \dot{Q}}{\partial T}\right)_\rho > 0$.

Physical Interpretation

Thermal instability reflects the competition between the gass ability to re-establish thermal balance via pressure work and the local response of heating and cooling to perturbations. Two characteristic timescales govern the dynamics:

$$t_{\text{cool}} \sim \frac{P_0}{(\gamma-1)\rho_0\,|\dot{Q}_T|}, \qquad t_{\text{sound}} \sim \frac{1}{kc_s}. \tag{11.72}$$

If $t_{\text{cool}} \ll t_{\text{sound}}$, the gas cools faster than it can adjust its pressure, leading to nearly isochoric behavior. Conversely, $t_{\text{cool}} \gg t_{\text{sound}}$ favors isobaric evolution, where pressure remains uniform.

In astrophysical systems, such as the interstellar medium, cooling laws often scale as $\Lambda \propto \rho^2 T^\alpha$. The Field criterion then becomes

$$\alpha < 1 \implies \text{isobaric instability},$$

so that cooling decreases too weakly with temperature to stabilize perturbations. The nonlinear outcome of this process is the formation of cold, dense condensations embedded in a warm, diffuse backgrounda defining feature of multi-phase astrophysical gases.

Big Idea

A fluid in local thermal balance becomes unstable if the local cooling rate increases with temperature at constant pressure:

$$\left(\frac{\partial \dot{Q}}{\partial T}\right)_P > 0.$$

Small perturbations then grow exponentially as radiative losses amplify temperature and density contrasts, leading to **spontaneous phase separation** of the gas into cold, dense and warm, diffuse components.

11.4 Convection

Part III

Kinetic Theory

12. Fundamentals of Kinetic Theory

Recall from our early conversation about fluid dynamics that the fluid approximation demands that the following conditions be satisfied for fluid elements:

1. **Sufficiently small:** The scale ℓ must be much smaller than the length scales over which any relevant quantity q varies appreciably. In other words, relative variations $\delta q/q$ across ℓ should satisfy

$$\delta q/q \ll 1 \tag{12.1}$$

   This ensures we can meaningfully associate a well-defined, approximately uniform value of q to each fluid element, neglecting internal fluctuations within that element.

In addition,

2. **Sufficiently statistical:** The scale ℓ must be large enough to contain many particles, so that microscopic, particle-level fluctuations are negligible compared to the collective, ensemble behavior. Formally, this requires the particle number density n to satisfy

$$\ell^3 n \gg 1. \tag{12.2}$$

3. **Thermal**: The scale ℓ must be much much larger than the mean free path to ensure that cells remain thermally equilibrated.

In some scenarios, we can satisfy the first two conditions, but not the third condition. In these scenarios, we have a so-called **collisionless-fluid**:

Definition 12.0.1 — Collisionless Fluid. A **collisionless fluid** is a continuous medium whose constituent particles interact primarily through longrange collective forces (e.g., gravity, electromagnetic fields), rather than through shortrange binary collisions. Formally, such a system satisfies the first two conditions of the fluid approximationstatistical

smoothness and local coherencebut fails the third:

$$\ell_{\text{mfp}} \gtrsim \ell,$$

where ℓ_{mfp} is the mean free path. In this regime, microscopic interactions are too infrequent to establish local thermodynamic equilibrium, and the evolution of the system can no longer be described by hydrodynamic equations closed by an equation of state. Instead, one must consider the evolution of the *phasespace distribution function* describing the full statistical state of the ensemble.

Because thermodynamic closure is lost, we can no longer describe the system solely through macroscopic quantities such as ρ, P, or $\mathbf{u}$. Instead, we must examine the evolution of the system in its full 6dimensional phase space. This is the domain of **kinetic theory**, which provides a statistical description of systems composed of a large number of particles interacting via prescribed microscopic laws.

12.1 Phase Space and the Distribution Function

Consider an ensemble of identical particles of mass m, whose individual states are specified by their positions $\mathbf{x}$ and (canonical) momenta $\mathbf{p}$. The set of all possible states forms a **phase space** Γ, which is a 6dimensional manifold equipped with canonical coordinates

$$\Gamma = \{(\mathbf{x}, \mathbf{p}) \mid \mathbf{x} \in \mathbb{R}^3, \mathbf{p} \in \mathbb{R}^3\}.$$

Each small element of phase space,

$$d^3\mathbf{x}\, d^3\mathbf{p},$$

represents the volume occupied by particles whose positions lie within $(\mathbf{x}, \mathbf{x}+d\mathbf{x})$ and whose momenta lie within $(\mathbf{p}, \mathbf{p}+d\mathbf{p})$.

We then define the **distribution function** $f(\mathbf{x}, \mathbf{p}, t)$ as the number density of particles in phase space:

Definition 12.1.1 — Distribution Function. The function $f(\mathbf{x}, \mathbf{p}, t)$ is defined such that

$$f(\mathbf{x}, \mathbf{p}, t)\, d^3x\, d^3p$$

gives the **number of particles** contained within the infinitesimal phasespace volume $d^3x\, d^3p$ centered at $(\mathbf{x}, \mathbf{p})$ at time t.

This is the fundamental quantity in statistical physics and will be the focus of the remainder of this chapter.

12.1.1 Moments of the Distribution

The distribution function $f(\mathbf{x}, \mathbf{p}, t)$ encodes the complete statistical state of a collisionless system. However, in practice, we are often interested in macroscopic quantities, such as

density, mean velocity, and pressure, that can be obtained as **moments** of f over momentum space. These moments serve as the bridge between kinetic theory and the familiar continuum fluid variables.

The **number density** of particles at position $\mathbf{x}$ and time t is obtained by integrating f over all momenta:

$$n(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{p}, t)\, d^3p.$$

If each particle has mass m, the corresponding **mass density** is

$$\rho(\mathbf{x}, t) = m\, n(\mathbf{x}, t) = m \int f(\mathbf{x}, \mathbf{p}, t)\, d^3p.$$

The zeroth moment thus provides the spatial density field, analogous to the $\rho(\mathbf{x}, t)$ used in hydrodynamics.

If we now move into the realm of first order moments, we can now construct various expectation values. The **mean velocity** (or bulk flow velocity) at each point is given by the momentumweighted average of the particle velocities:

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int \mathbf{v}\, f(\mathbf{x}, \mathbf{p}, t)\, d^3p, \qquad \mathbf{v} = \frac{\mathbf{p}}{m}.$$

This vector field represents the macroscopic velocity of the fluid element located at $\mathbf{x}$. The total momentum density of the system is therefore

$$\rho\mathbf{u} = \int \mathbf{p}\, f(\mathbf{x}, \mathbf{p}, t)\, d^3p.$$

Finally, we can scale up our conversation to second-order moments. The first of these is the **velocity dispersion tensor** quantifies the random or thermal motions about this mean:

$$\sigma_{ij}^2(\mathbf{x}, t) = \langle (v_i - u_i)(v_j - u_j) \rangle = \frac{1}{n(\mathbf{x}, t)} \int (v_i - u_i)(v_j - u_j)\, f(\mathbf{x}, \mathbf{p}, t)\, d^3p.$$

This symmetric tensor encodes the anisotropy of microscopic motions at each point in the system. Its trace defines the mean square speed relative to the bulk flow:

$$\langle v'^2 \rangle = \operatorname{Tr} \boldsymbol{\sigma}^2 = \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2.$$

In an isotropic system, the tensor reduces to

$$\sigma_{ij}^2 = \sigma^2\, \delta_{ij},$$

and one may define a scalar pressure through

$$P = \rho\, \sigma^2.$$

This recovers the familiar thermodynamic relation $P = \rho k_B T/m$ in the Maxwellian limit, where $\sigma^2 = k_B T/m$.

The hierarchy of velocity moments may be extended to higher order. The second moment described the dispersion or pressure tensor, which characterizes the internal stresses

within the collisionless fluid. The next in the sequence, the **third moment**, encodes the *flux of random kinetic energy* carried by microscopic motions.

The **third-order moment** of the distribution function is defined as

$$Q_{ijk}(\mathbf{x},t) = \int (v_i - u_i)(v_j - u_j)(v_k - u_k)\, f(\mathbf{x},\mathbf{p},t)\, d^3p.$$

This fully symmetric rank3 tensor describes the transport of the velocitydispersion tensor itself: i.e., the rate at which random kinetic energy is advected through the system. In fluid dynamics, the contraction of Q_{ijk} with the velocity field leads to the familiar **heat flux vector**:

$$q_i(\mathbf{x},t) = \frac{1}{2}\,\rho\,\langle v'^2 (v_i - u_i)\rangle = \frac{1}{2}\,m \int v'^2 (v_i - u_i)\, f(\mathbf{x},\mathbf{p},t)\, d^3p.$$

Here $v_i' = v_i - u_i$ represents the peculiar (random) velocity relative to the local mean flow. The vector $\mathbf{q}$ quantifies the flow of internal energy within the fluid elementanalogous to the conductive or radiative heat flux in collisional systems.

Summary of the Moment Hierarchy.

We may summarize the first three velocity moments of the distribution function as follows:

$$\boxed{\begin{aligned}
\rho(\mathbf{x},t) &= m \int f\, d^3p, \\
\rho\, u_i(\mathbf{x},t) &= \int p_i\, f\, d^3p, \\
\rho\, \sigma_{ij}^2(\mathbf{x},t) &= \int (v_i - u_i)(v_j - u_j)\, f\, d^3p, \\
Q_{ijk}(\mathbf{x},t) &= \int (v_i - u_i)(v_j - u_j)(v_k - u_k)\, f\, d^3p.
\end{aligned}}$$

Each successive moment describes a higher level of structure in the velocity distribution and introduces new tensor quantities: the *density* and *bulk motion* at first order, the *stress (pressure) tensor* at second order, and the *energy flux tensor* at third order. In kinetic theory, this hierarchy continues indefinitely, and the problem of finding a finite, closed set of equations for the macroscopic variables is known as the **moment closure problem**.

12.2 The Dynamics of Collisionless Fluids

To describe the time evolution of a collisionless system, we now introduce the dynamical law governing $f(\mathbf{x},\mathbf{p},t)$. Each individual particle follows a trajectory in phase space determined by Hamiltons equations for a given singleparticle Hamiltonian $H(\mathbf{x},\mathbf{p},t)$:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}.$$

The total time derivative of the distribution function along a particles trajectory is therefore

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla_{\mathbf{x}} f + \frac{d\mathbf{p}}{dt}\cdot\nabla_{\mathbf{p}} f.$$

12.2.1 Liouvilles Theorem

A key result of Hamiltonian dynamics is that the phasespace volume occupied by a set of trajectories is conserved in time. Formally, the flow in phase space is *incompressible*, satisfying

$$\nabla_{\mathbf{x}} \cdot \dot{\mathbf{x}} + \nabla_{\mathbf{p}} \cdot \dot{\mathbf{p}} = 0.$$

Therefore, for a collisionless ensemble of particles,

Theorem 12.2.1 — Liouvilles Theorem. The phasespace density $f(\mathbf{x}, \mathbf{p}, t)$ of a collisionless system is conserved along trajectories in phase space:

$$\frac{df}{dt} = 0.$$

This expresses the fact that, in the absence of collisions, the number of particles contained in a moving volume element of phase space remains constant as it is advected by the Hamiltonian flow.

12.2.2 The Collisionless Boltzmann (Vlasov) Equation

Expanding the total derivative in Liouvilles theorem gives the **Collisionless Boltzmann Equation** (CBE), also known as the **Vlasov equation**:

$$\boxed{\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{p}} f = 0,}$$

where $\Phi(\mathbf{x}, t)$ is the gravitational (or more generally, meanfield) potential appearing in the singleparticle Hamiltonian,

$$H(\mathbf{x}, \mathbf{p}) = \frac{p^2}{2m} + m\Phi(\mathbf{x}, t).$$

This equation describes the conservation of phasespace density under the combined effects of free streaming and collective forces. It forms the foundation of **stellar dynamics**, **galactic dynamics**, and the **collisionless plasma** theory.

Integrating the CBE over momentum space reproduces the macroscopic continuity and momentum equations of fluid dynamics, but without the assumption of local thermodynamic equilibrium. In this way, the collisionless Boltzmann equation generalizes hydrodynamics to systems where microscopic collisions are negligible but collective fields remain dynamically important.

12.2.3 Collisionless Continuity Equation

We begin from the (gravitational) Collisionless Boltzmann Equation (CBE)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{v}} f = 0, \qquad (\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{12.3}$$

Define the number density and bulk velocity by

$$n(\mathbf{x}, t) \equiv \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{v}, t)\, d^3v, \qquad \mathbf{u}(\mathbf{x}, t) \equiv \frac{1}{n} \int_{\mathbb{R}^3} \mathbf{v}\, f\, d^3v,$$

and the mass density $\rho \equiv mn$ for particle mass m.

Integrate (12.3) over all velocities. We treat each term carefully.

First Term:

Because the **v**integration does not act on t,

$$\int \frac{\partial f}{\partial t}\, d^3v = \frac{\partial}{\partial t}\int f\, d^3v = \frac{\partial n}{\partial t}.$$

Second Term:

For the second term, we have

$$\int \mathbf{v}\cdot\nabla_{\mathbf{x}} f\; d^3\mathbf{v}.$$

Since $\mathbf{v}$ is independent of $\mathbf{x}$, we can pull out the covariant derivative to get

$$\int \mathbf{v}\cdot\nabla_{\mathbf{x}} f\; d^3\mathbf{v} = \nabla_{\mathbf{x}}\cdot\int \mathbf{v} f\; d^3\mathbf{v}.$$

This is precisely the average velocity, so we have

$$\int \mathbf{v}\cdot\nabla_{\mathbf{x}} f\; d^3\mathbf{v} = \nabla_{\mathbf{x}}\cdot(n\mathbf{u}).$$

Third Term:

If we now also integrate over the third term, we have

$$-\int \nabla_{\mathbf{x}}\Phi\cdot\nabla_{\mathbf{p}} f = -\nabla_{\mathbf{x}}\Phi\cdot\int \nabla_{\mathbf{p}} f\; d^3\mathbf{v}.$$

This last term most go to on the basis of divergence theorem. As such, we have the **continuity equation**:

$$\boxed{\frac{\partial n}{\partial t} + \nabla_{\mathbf{x}}\cdot(n\mathbf{u}) = 0.} \tag{12.4}$$

Equivalently, we have an equation for the density which is precisely the same as that arising in the fluid theory:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0.$$

12.2.4 The Jeans Equations

We now derive the **Jeans equations** by taking the first velocity moment of the Collisionless Boltzmann Equation (CBE). Starting from

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}}\Phi\cdot\nabla_{\mathbf{v}} f = 0,$$

we multiply through by $\mathbf{v}$ and integrate over all velocities to obtain the evolution of the mean momentum field. This procedure gives the collisionless analogue of the momentum (Euler) equation in hydrodynamics.

We begin by recalling the definitions of the zeroth and first velocity moments:

$$n(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)\, d^3v, \qquad \mathbf{u}(\mathbf{x},t) = \frac{1}{n}\int \mathbf{v} f(\mathbf{x},\mathbf{v},t)\, d^3v,$$

and define the second moment (velocitydispersion tensor)

$$\sigma_{ij}^2 = \frac{1}{n}\int (v_i - u_i)(v_j - u_j)\, f\, d^3v.$$

These quantities will naturally appear as we take successive moments of the CBE.

First Term:
The timederivative term gives

$$\int \mathbf{v}\,\frac{\partial f}{\partial t}\,d^3v = \frac{\partial}{\partial t}\int \mathbf{v} f\,d^3v = \frac{\partial(n\mathbf{u})}{\partial t}.$$

Second Term:
The streaming term involves advection of momentum through space. We have, in components,

$$\int v^\mu v^\nu \nabla_\nu f\; d^3v = \nabla_\nu \cdot \int v^\mu v^\nu f\; d^3v.$$

This is a **second moment** of the distribution and should be connected to the dispersion tensor. We also know that we have

$$\int (v^\nu - u^\nu)(v^\mu - u^\mu) f\; d^3v = \int v^\nu v^\mu f\; d^3v - \int u^\nu u^\mu f\; d^3v.$$

As such, we have the term

$$\nabla_{\mathbf{x}} \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (n\boldsymbol{\sigma}^2).$$

Third Term:
Finally, consider the gravitational term. Multiplying by $\mathbf{v}$ and integrating gives

$$-\int \mathbf{v}\,\nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{v}} f\,d^3v.$$

We integrate by parts in velocity space. Since $\nabla_{\mathbf{v}}\mathbf{v} = \mathbf{I}$ and $f \to 0$ as $|\mathbf{v}| \to \infty$, the boundary term vanishes, and we find

$$-\int \mathbf{v}\,\nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{v}} f\,d^3v = \int f\,(\nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{v}}\mathbf{v})\,d^3v = n(-\nabla_{\mathbf{x}}\Phi).$$

Combining all three terms, we obtain the momentum equation in numberdensity form:

$$\frac{\partial(n\mathbf{u})}{\partial t} + \nabla_{\mathbf{x}} \cdot \big(n\mathbf{u}\mathbf{u}\big) + \nabla_{\mathbf{x}} \cdot \big(n\boldsymbol{\sigma}^2\big) + n\nabla_{\mathbf{x}}\Phi = 0.$$

Multiplying through by m (so that $\rho = mn$) gives the **collisionless Jeans momentum equation** in conservative form:

$$\boxed{\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla_{\mathbf{x}} \cdot \big(\rho\mathbf{u} \otimes \mathbf{u}\big) = -\nabla_{\mathbf{x}} \cdot \big(\rho\,\boldsymbol{\sigma}^2\big) - \rho\,\nabla_{\mathbf{x}}\Phi.} \tag{12.5}$$

Equivalently

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla_{\mathbf{x}}(\mathbf{p} \otimes \mathbf{u}) = -\nabla \cdot (\rho\boldsymbol{\sigma}^2) - \rho\nabla_{\mathbf{x}}\Phi.$$

Equation (12.5) is the collisionless analogue of the Euler momentum equation. The first two terms describe temporal and convective changes in momentum density, the third represents

the divergence of the *velocitydispersion tensor* (an anisotropic stress arising from random particle motions), and the final term is the gravitational acceleration. In an isotropic system, $\boldsymbol{\sigma}^2 = \sigma^2 \mathbf{I}$ and the equation reduces to

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla P - \rho \nabla \Phi, \qquad P = \rho \sigma^2,$$

which is the familiar Euler equation for a selfgravitating fluid. In the general, anisotropic case, however, $\rho \boldsymbol{\sigma}^2$ acts as a full **stress tensor**, and no simple closure exists without higherorder velocity moments. This lack of closure defines the **moment hierarchy problem** in kinetic theory.

12.2.5 The Tensor Virial Theorem

13. Solutions of the Boltzmann Equation

13.1 The Jeans' Theorem and Eddington Formula

13.1.1 The Jeans' Theorem

- Introduce the concept and then state the theorem directly. - Desribe the relevance.

13.1.2 The Eddington Formula

- Introduce the spherical case and perform the derivation for the isotropic scenario.

13.2 Schwarzchild's Method

14. Stochastic Processes

Part IV

Plasma Physics

Part V

Phenomenology

Bibliography

G. B. Field. Thermal Instability. *ApJ*, 142:531, Aug. 1965. doi: 10.1086/148317.

N. I. Shakura and R. A. Sunyaev. Black holes in binary systems. Observational appearance. *Astronomy and Astrophysics*, 24:337–355, Jan. 1973.